BHARATHIDASANAR MATRIC HIGHER SECONDARY SCHOOL ARAKONAM

XII –MATHEMATICS

MATERIAL – 7 &8 UNIT (6 MARKS & 10 MARKS)

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INTEGRAL CALCULUS AND ITS APPLICATIONS

Ten mark questions:

1. Find the area between the curves $y = x^2 - x - 2$, x-axis and the lines $x = -2$ and $x = 4$

Solution :

$y = x^2 - x - 2$

$= (x + 1)(x - 2)$

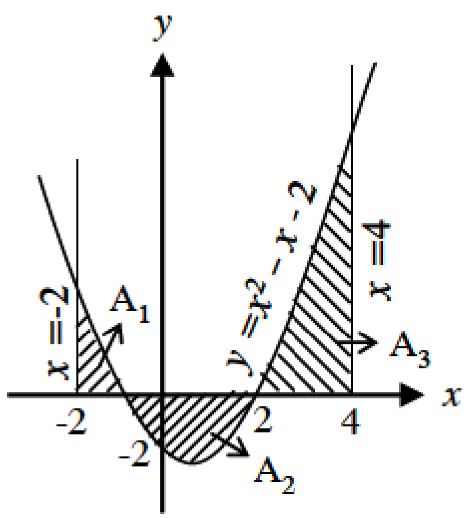
This curve intersects x-axis at $x = -1$ and $x = 2$

Required area = $A1 + A2 + A3$

The part $A2$ lies below x-axis.

$\therefore A_2 = -\int_{-1}^{2} y\,dx$

Hence required area



$$A = \int_{-2}^{-1} y\,dx + \int_{-1}^{2} (-y)\,dx + \int_{2}^{4} y\,dx$$

$$= \int_{-2}^{-1} (x^2 - x - 2)dx + \int_{-1}^{2} -(x^2 - x - 2)\,dx + \int_{2}^{4} (x^2 - x - 2)dx$$

$$= \frac{11}{6} + \frac{9}{2} + \frac{26}{3}$$

$= 15$ sq. units

2. Find the area between the line $y = x + 1$ and the curve $y = x^2 - 1$.

Solution :

To get the points of intersection of the curves we should solve the equations

$y = x + 1$ and $y = x^2 - 1$.

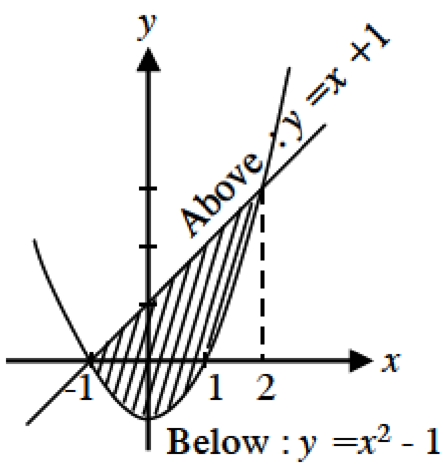
we get, $x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$\Rightarrow (x - 2)(x + 1) = 0$

$\therefore x = -1$ or $x = 2$

$\therefore$ The line intersects the curve at $x = -1$ and $x = 2$.



Required area = $\int_a^b \{f(x)[above] - g(x)[below]\}$ dx

$= \int_{-1}^{2} [(x + 1) - (x^2 - 1)]dx$

$= \int_{-1}^{2} [2 + x + x^2]$ dx

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$

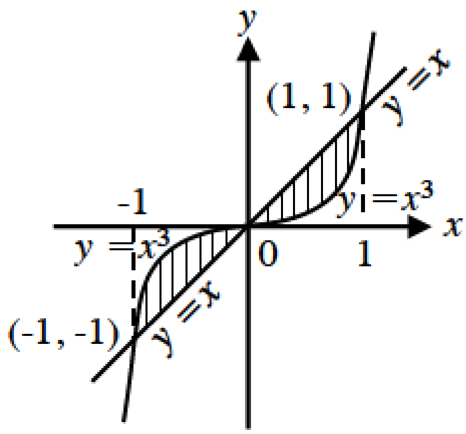
$= [4 + 2 - \frac{8}{3}] - [-2 + \frac{1}{2} + \frac{1}{3}]$

$= \frac{9}{2}$ sq.units

3. Find the area bounded by the curve $y = x^3$ and the line $y = x$.

Solution :

The line $y = x$ lies above the curve $y = x^3$ in the first quadrant and $y = x^3$ lies above the line $y = x$ in the third quadrant. To get the points of intersection, solve the curves $y = x^3$, $y = x \Rightarrow x^3 = x$. We get $x = \{0, \pm 1\}$



The required area = $A_1 + A2 = \int_{-1}^{0} [g(x) - f(x)]\, dx + \int_{0}^{1} [f(x) - g(x)]\, dx$

$$= \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} (x - x^3)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \left(0 - \frac{1}{4}\right) - \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) - \left(\frac{1}{4} - 0\right)$$

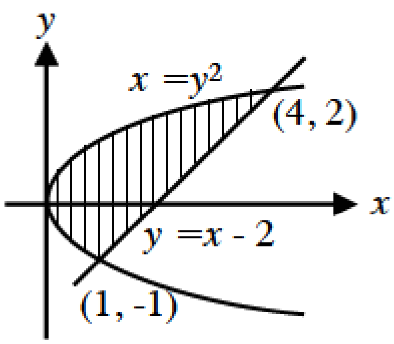
$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{2}\ sq.units$$

4. Find the area of the region enclosed by $y^2 = x$ and $y = x - 2$

Solution :

The points of intersection of the parabola $y^2 = x$ and the line $y = x - 2$ are $(1, -1)$ and $(4, 2)$



To compute the region [shown in figure] by integrating with respect to x, we would have to split the region into two parts, because the equation of the lower boundary changes at $x = 1$. However if we integrate with respect to y no splitting is necessary.

Required area = $\int_{-1}^{2}[f(y) - g(y)]\, dy$

$$= \int_{-1}^{2}[(y+2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

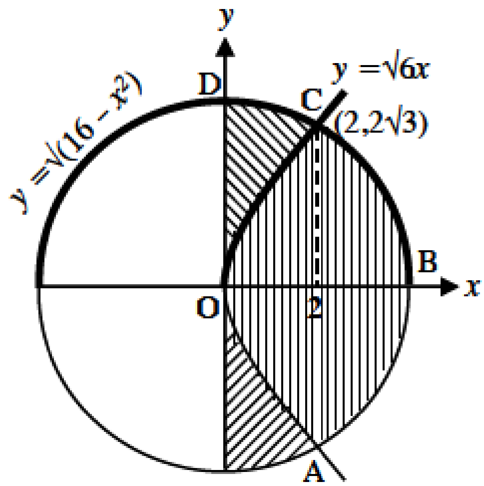
$$= \left(\frac{4}{2} - \frac{1}{2}\right) + (4+2) + \left(\frac{8}{3} + \frac{1}{3}\right)$$

$$= \frac{3}{2} + - \frac{9}{3} = \frac{9}{2} \text{ sq.units}$$

5. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$

Solution :

The points of intersection of $x^2 + y^2 = 16$ and $y^2 = 6x$ are $(2, \ 2\sqrt{3})$ and $(2, \ -2\sqrt{3})$



Required area is *OABC* Due to symmetrical property,

The required area *OABC* = 2 *OBC*

i.e., 2{[Area bounded by $y^2 = 6x, x = 0$, $x = 2$ and x-axis] + [Area bounded by $x^2 + y^2 = 16$, $x = 2$, $x = 4$and x-axis]}

$$\text{Required area} = 2\int_0^2 \sqrt{6x}\ \ dx + 2\int_2^4 \sqrt{16 - x^2}\, dx$$

$$= 2\sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$= \frac{8\sqrt{12}}{3} - 2\sqrt{12} + 8\pi - \frac{8\pi}{3}$$

$$= \frac{4}{3}(4\pi + \sqrt{3})\ \text{sq.units.}$$

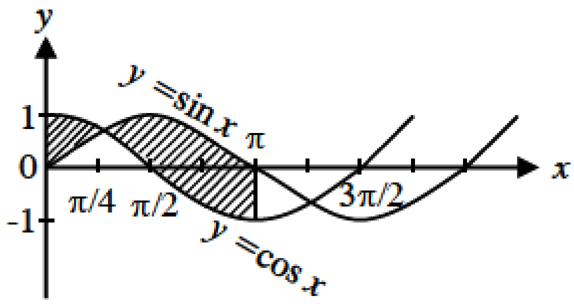
6. Compute the area between the curve $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

Solution :

To find the points of intersection solve the two equations.

$\text{Sin } x = \cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}$

$\text{Sin } x = \cos x = -\frac{1}{\sqrt{2}} \Rightarrow x = \frac{5\pi}{4}$



From the figure we see that cos x > sin x for $0 \leq x < \frac{\pi}{4}$ and sin x > cos x for $\frac{\pi}{4} < x < \pi$

$\therefore$ Area $A = \int_0^{\pi/4}(\cos x\, dx - \sin x\, dx) + \int_{\pi/4}^{\pi}(\sin x - \cos x)dx$

$= [\sin x + \cos x]_0^{\pi/4} + [(-\cos x - \sin x]_{\pi/4}^{\pi}$

$= (\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) - (\sin 0 + \cos 0) + (-\cos\pi - \sin\pi) - (-\cos\frac{\pi}{4} - \sin\frac{\pi}{4})$

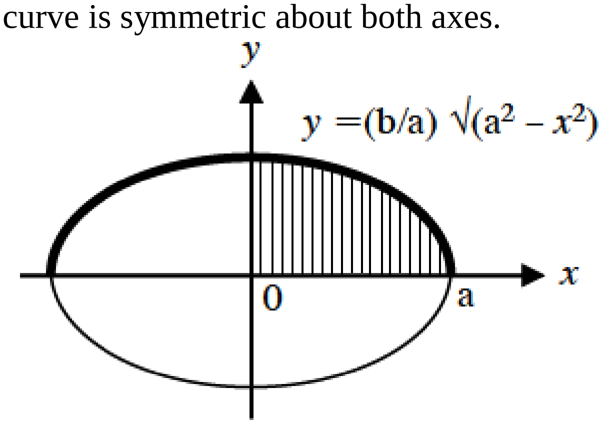
$= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}})$ - (0 +10 +(1- 0) –($-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$)

$= 2\sqrt{2}$ sq.units .

7. Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution :

The curve is symmetric about both axes.



$\therefore$ Area of the ellipse = 4 $\times$ Area of the ellipse in the I quadrant.

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,\mathrm{dx}$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,\mathrm{dx}$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0\right]$$

$$= \pi ab\ sq.units.$$

8. Find the area of the curve $y^2 = (x-5)^2(x-6)$(i) between $x = 5$ and $x = 6$
(ii) between $x = 6$ and $x = 7$

Solution :

(i) $y^2 = (x-5)^2(x-6)$

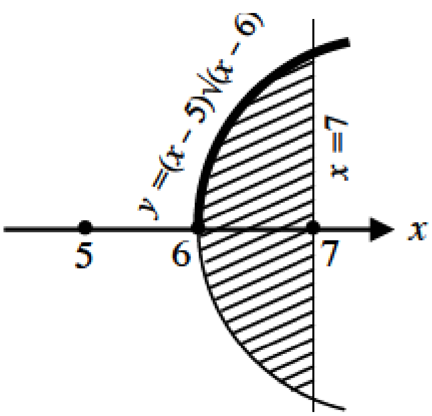
$\therefore y = (x-5)\ x-6$

This curve cuts the x-axis at $x = 5$ and at $x = 6$

When x takes any value between 5 and 6, y^2 is negative.

$\therefore$ The curve does not exist in the interval $5 < x < 6$.

Hence the area between the curve at $x = 5$ and $x = 6$ is zero



(ii) Required area = $\int_a^b y\,dx$

$$= 2\int_6^7 (x-5)\sqrt{x-6}\,dx$$

(Since the curve is symmetrical about x- axis)

Take t = x – 6

dt = dx

x	*6*	*7*
t	*0*	*1*

$= 2\int_0^1 (t+1)\,dx$

$=2\int_0^1 (t^{3/2} + t^{1/2})$ dt

$= 2\left[\frac{t^{5/2}}{5/2} - \frac{t^{1/2}}{3/2}\right]_0^1$

$= 2\left(\frac{2}{3} + \frac{2}{3}\right)$

$= 2\left(\frac{6+10}{15}\right) = \frac{32}{15}$ sq. units

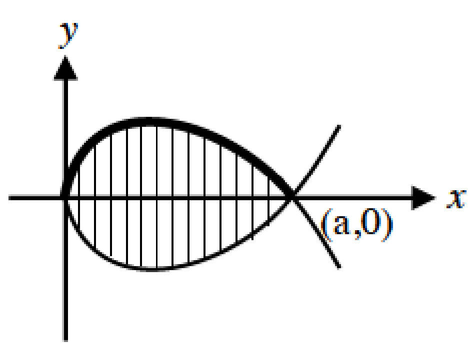
9. Find the area of the loop of the curve $3ay^2 = x(x-a)^2$

Solution :

Put $y = 0$; we get $x = 0$,

It meets the x-axis at $x = 0$ and $x = a$

∴Here a loop is formed between the points (0, 0) and (a, 0) about x-axis. Since the curve is symmetrical about x-axis, the area of the loop is twice the area of the portion above the x-axis.



Required area $= 2\int_0^a y\,dx$

$$= -2\int_0^a \frac{\sqrt{x}(x-a)}{\sqrt{3a}}\,dx$$

$$= -\frac{2}{\sqrt{3a}}\int_0^a [x^{3/2} - a\sqrt{x}\,]\,dx$$

$$= -\frac{2}{\sqrt{3a}}\left[\frac{2}{5}x^{5/2} - \frac{2a}{3}x^{3/2}\right]_0^a$$

$$= \frac{8a^2}{15\sqrt{3}}$$

$$= \frac{8\sqrt{3}a^2}{45}$$

Required area $= \frac{8\sqrt{3}a^2}{45}$ sq. units

10. Find the area bounded by x-axis and an arch of the cycloid $x = a\,(2t - \sin 2t)$, $y = a\,(1 - \cos 2t)$

Solution :

The curves crosses x-axis when $y = 0$.

$\therefore a(1 - \cos 2t) = 0$

$\therefore \cos 2t = 1$; $2t = 2n\pi,\ n \in z$

$\therefore t = 0, \pi, 2\pi, \cdots$

$\therefore$ One arch of the curve lies between 0 and π

Required area $= \int_a^b y\,dx$

$= \int_0^\pi a(1 - \cos 2t)\, 2a\,(1 - \cos 2t)\,dt$

$y = a(1 - \cos 2t)$; $x = a\,(2t - \sin 2t)$

$dx = 2a(1 - \cos 2t)\,dt$

$= 2a^2\int_0^\pi (1 - \cos 2t)^2\,dt$

$= 8a^2\int_0^{\pi} sin^4 t$ dt

$= 2$ x $8a^2 \int_0^{\pi/2} sin^4 t$ dt $\left(\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)dx\right)$

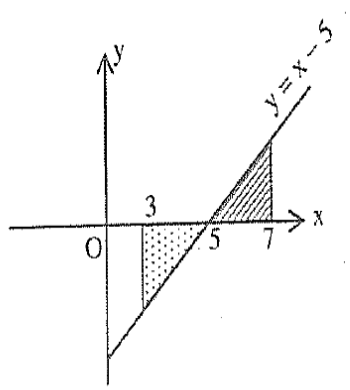
$= 16a^2\left[\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$

$= 3\,\pi a^2\ sq.units$

11. Find the area of the region bounded by the curve y = $3x^2$ - x and the x- axis between x = -1 and x = 1

Solution:

Draw the parabola y = $3x^2$– x



It is open upward and meets the x- axis at x = 0 and x = 1/3 (put x=0)

The required area is the combination of three pieces. They are

(i)Bounded by the curve , x = -1 , x=0 and x-axis . it lies above x-axis and area of this part is $\int_{-1}^{0} y\,dx$

(ii)Bounded by the curve , x =0 , x= 1/3 and x- axis . it lies below x-axis and hence the area of this part is $\int_0^{1/3}(-y)dy$

(iii)Bounded by the curve , x= 1/3 , x=1 and x-axis . it lies above x-axis and hence the area of this part is $\int_{1/3}^{1} ydx$

Sum of the three parts = $\int_{-1}^{0}(3x^2 - x)dx + \int_0^{1/3}(x - 3x^2)dy + \int_{1/3}^{1}(3x^2 - x)\,dx$

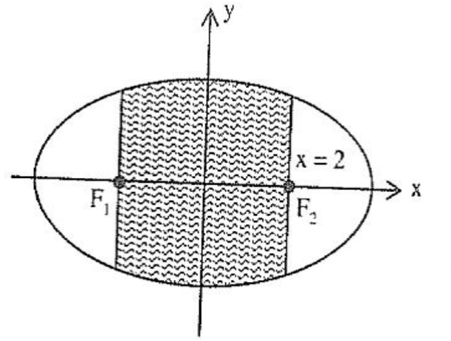
$$=\left[x^3 - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - x^3\right]_0^{1/3} + \left[x^3 - \frac{x^2}{2}\right]_{1/3}^{1}$$

$$= \left[0 - (-1 - \tfrac{1}{2})\right] + \left[(\tfrac{1}{18} - \tfrac{1}{27}) - 0\right] + \left[(1 - \tfrac{1}{2}) - (\tfrac{1}{27} - \tfrac{1}{18})\right]$$

$$= \frac{55}{27} \text{ sq.units}$$

12. Find the area of the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ between the two latus rectums.

Solution:



Equations of the latus rectums are $x = \pm\, ae$

$a^2 = 9,\ b^2 = 5 \implies e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{2}{3}$

ae=2

Thus the equations of L.R are $x = \pm 2$. The required area is bounded by the ellipse and x = - 2 , x = 2

Since the curve is symmetrical about both axes, the required area is 4 times the area in the first quadrant. I.e., the area bounded by the curve $\frac{x^2}{9} + \frac{y^2}{5} = 1$ or $y = \frac{\sqrt{5}}{3}\sqrt{9 - x^2}$, x= 0 , x= 2 and x-axis.

Required area = $4\int_0^2 y\, dx$

$$= \int_0^2 \frac{\sqrt{5}}{3} . \sqrt{9 - x^2}\, dx$$

$$= \frac{4\sqrt{5}}{3}\left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2}\sin^{-1}(\tfrac{x}{3})\right]_0^2$$

$$= \frac{4\sqrt{5}}{3}\ [\sqrt{5} + \tfrac{9}{2}\sin^{-1}(\tfrac{x}{3})]$$

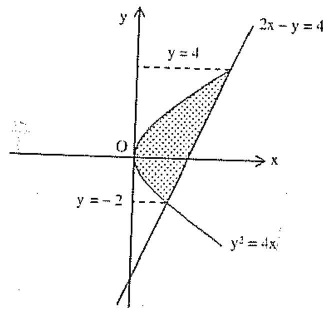
13. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $2x - y = 4$.

Solution:

To find the limits solve the two equations

i.e., $y^2 = 4x$ and $2x - y = 4$.

i.e., y = -2, 4



Note that , it is difficult to find the required area by using x-axis. Use y-axis to find the area

Required area = $\int_{-2}^{4} (x_1 - x_2)\, dy$

Again note that the line 2x – y = 4 gives the maximum area with the y-axis when compare with the parabola and hence consider the line as first curve and the parabola as the second curve.

i.e., x_1 means x from the line.

x_2 means x from the parabola.

$$A = \int_{-2}^{4} \left[\frac{y+4}{2} - \frac{y^2}{4}\right] dy$$

$$= \left[\frac{1}{2}\left(\frac{y^2}{2} + 4y\right) - \frac{y^3}{12}\right]_{-2}^{4}$$

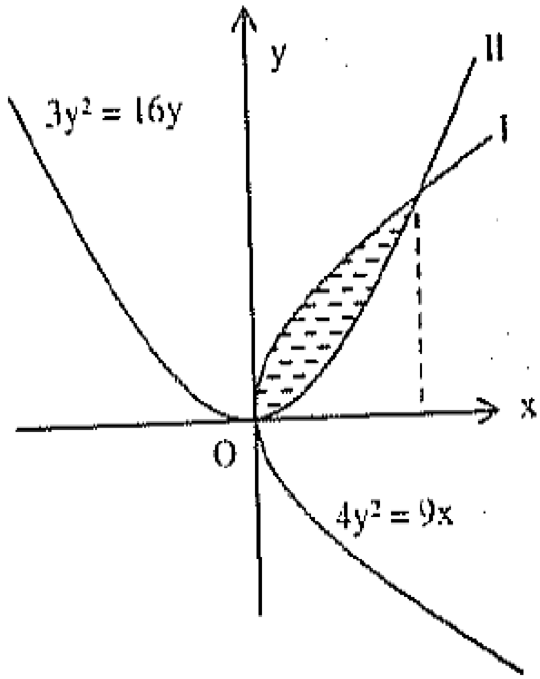
$$= \left[\frac{1}{2}\left(\frac{16}{2} + 16\right) - \frac{64}{12}\right] - \left[\frac{1}{2}\left(\frac{4}{2} - 8\right) + \frac{8}{12}\right]$$

= 9 sq.units.

14. Find the common area enclosed by the parabolas $4y^2 = 9x$ and $3x^2 = 16y$\

Solution:

Solving the equations $4y^2$= 9x and $3x^2$= 16y , we get the point of intersections as (0,0) and (4,3)



[note: we can solve this problem either by using x-axis (i.e.,x = 0, x = 4) or by y-axis,

$4y^2$= 9x gives the maximum area and $3x^2$= 16y gives the minimum area.thus

w.r. to x-axis $4y^2$= 9x is the first curve and $3x^2$= 16y is the second curve. but if we

take y-axis as axis of bondedness, $3x^2$= 16y is the first curve and $4y^2$= 9x is

the second curve.

We solve this by using x-axis.

Required area =$\int_0^4 (y_1 - y_2)\, dx$

$= \int_0^4 \left[\frac{3}{2} x^{1/2} - \frac{3}{16} x^2\right] dx$

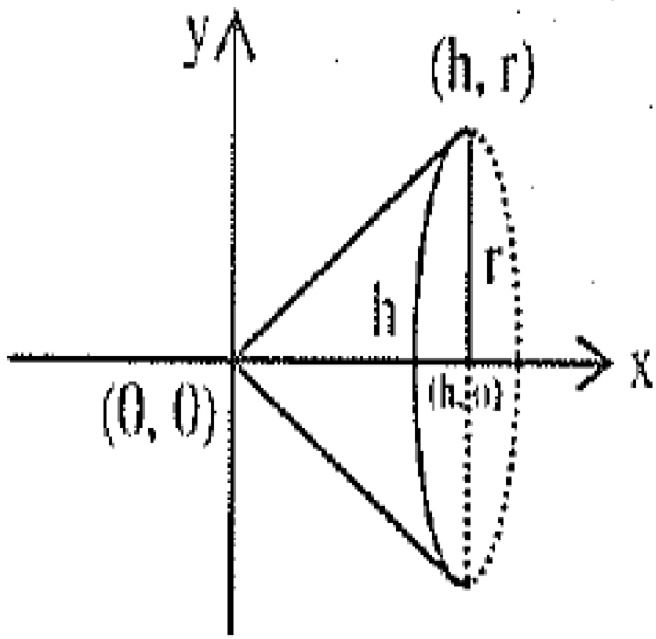
$= \left[x^{3/2} - \frac{x^3}{16}\right]_0^4$

= (8 - 4) – 0 =4 sq.units.

Note: the integral using y – axis is $\int_0^3 (x_1 - x_2)\, dy$

15. Derive the formula for the volume of a right circular cone with radius *'r'* and height *'h'*.

Solution:



To find the volume of the cone with base radius r and height h, revolve the area

of a triangle whose vertices are (0,0), (h,r) is y = $\frac{r}{h}x$

$\therefore$ The volume of the cone is obtained by revolving the area bounded by

y = $\frac{r}{h}x$ x= 0, x = h and x-axis, about x-axis.

i.e., V = $\pi\int_0^h y^2\,dx$

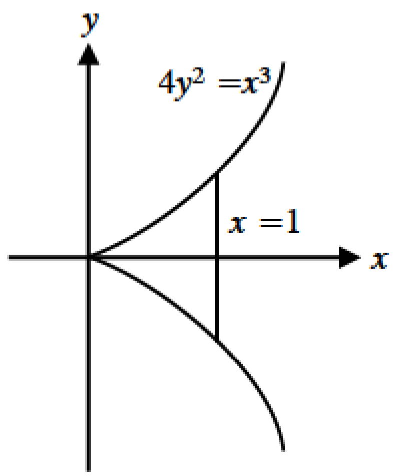
= $\pi\int_0^h \frac{r^2}{h^2}y^2\,dx$

= $\frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$

= $\frac{\pi r^2}{h^2}\left[\frac{h^3}{3}\right]$ =$\frac{1}{3}\pi r^3 h$ cub.units

16. Find the length of the curve $4y^2 = x^3$ between $x = 0$ and $x = 1$

Solution:



$4y^2 = x^3$

Differentiating with respect to x

$8y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{8y}$

$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + \frac{9x^4}{64y^2}}$

$= \sqrt{1 + \frac{9x^4}{16x^3}} = \sqrt{1 + \frac{9x}{16}}$

The curve is symmetrical about x-axis.

The required length $L = 2\int_0^1 \sqrt{1 + (\frac{dy}{dx})^2}\ dx$

$= 2\int_0^1 (1 + \frac{9x}{16})^{1/2} dx$

$= 2\left[\frac{(1+\frac{9x}{16})^{3/2}}{\frac{9}{16} \times \frac{3}{2}}\right]_0^1$

$= \frac{64}{27}\left[\frac{125}{64} - 1\right] = \frac{61}{27}$

17. Find the length of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$

Solution :

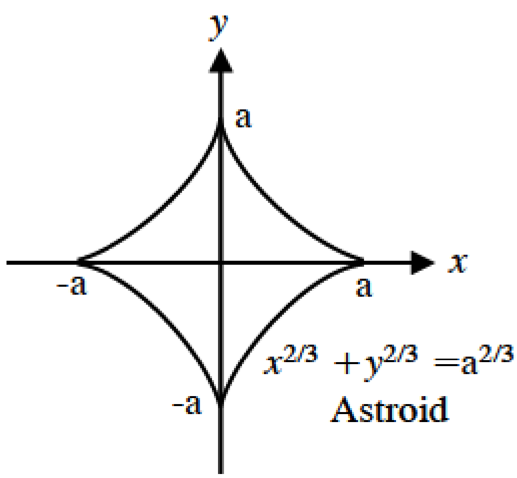
$x = a\cos^3 t$, $y = \sin^3 t$ is the parametric form of the given asteroid,

where $0 \le t \le 2\pi$

$\frac{dx}{dt} = -3a\cos^2 t \sin t$;

$\frac{dy}{dt} = 3a\sin^2 t \cos t$;

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = 3a\,sint\,cost$$



Since the curve is symmetrical about both axes, the total length of the curve is 4 times the length in the first quadrant.

But t varies from 0 to $\frac{\pi}{2}$ in the first quadrant

$$\text{Length of the entire curve} = 4\int_0^{\pi/2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$$

$$= 4\int_0^{\pi/2} 3a\sin t\cos t\,dt$$

$= 6a \int_0^{\pi/2} sin2\,t\,dt$

$= 6a \left[-\frac{cos\,2t}{2}\right]_0^{\pi/2}$

$= -3a[\cos\pi - cos\,0]$

$= -3a\,[-1 - 1\,]$

$= 6a$

18. Show that the surface area of the solid obtained by revolving the arc of the Curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x-axis is $2\pi\,[\sqrt{2} + \log(1 + \sqrt{2})]$

Solution :

$y = \sin x$

Differentiating with respect to x

$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + cos^2 x}$

Surface area = $\int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2}\;dx$

When the area is rotated about the x-axis.

$S = \int_1^{-1} 2\pi y\,\sqrt{1 + cos^2 x}\;dx$

Put $t = \cos x$

$dt = -\sin x\,dx$

x	0	π
t	1	-1

$= \int_1^{-1} 2\pi\sqrt{1 + t^2}(-dt)$

$= 4\pi \int_0^1 \sqrt{1 + t^2}\;dt$

$= 4\pi \left[\frac{t}{2}\sqrt{1 + t^2} + \frac{1}{2}\,log(t + \sqrt{1 + t^2}\right]_0^1$

$= 2\pi[\sqrt{2} + \log(1 + \sqrt{2}) - 0$

$= 2\pi[\sqrt{2} + \log(1 + \sqrt{2})$

19. Find the surface area of the solid generated by revolving the cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$ about its base (x-axis).

Solution :

$y = 0 \Rightarrow 1 + \cos t = 0 \Rightarrow \cos t = -1 \Rightarrow t = -\pi, \pi$

$x = a(t + \sin t)$; $y = a(1 + \cos t)$

$\frac{dx}{dt} = a(1 + \cos t)$; $\frac{dy}{dt} = -a \sin t$

$$\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \sqrt{(a^2(1+\cos t)^2 + a^2\sin^2 t)} = 2a\cos\frac{t}{2}$$

$$Surface\ area = \int_{-\pi}^{\pi} 2\pi a(1 + \cos t)\, 2a\cos\frac{t}{2}\, dt$$

$$= \int_{-\pi}^{\pi} 2\pi a \,.\, 2\cos^2\frac{t}{2} \,.\, 2a\cos\frac{t}{2}\, dt$$

$$= 16\pi a^2 \int_0^{\pi} \cos^3\frac{t}{2}\, dt$$

$$= 16\pi a^2 \int_0^{\pi/2} 2\cos^3 x\, dx \qquad \left[Take\, x = \frac{t}{2}\right]$$

$$= 32\pi a^2 I_3$$

$$= 32\pi a^2\, x\frac{2}{3}$$

$$= \frac{64}{3}\pi a^2 \text{ sq.units}$$

20. Find the perimeter of the circle with radius a.

Solution:

Take the circle with the centre as (0,0).

$\therefore$ the equation of the circle is $x^2 + y^2 = a^2$

The perimeter of the length is 4 times the length of the arc

Of the circle in the first quadrant between x = 0 and x = a

Perimeter = $4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Now, $x^2 + y^2 = a^2$

=> $2x + 2y\frac{dy}{dx} = 0$

=> $\frac{dy}{dx} = \frac{-x}{y}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}} = \frac{a}{y} = \frac{a}{\sqrt{a^2-x^2}}$

Perimeter = $4\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx$

$= 4a\left[sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= 4a\ [sin^{-1} 1 - 0]$

$= 4a \times \frac{\pi}{2}$

$= 2\pi a$

21. Find the length of the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$ between $t = 0$ and π.

Solution:

The period of one arc is 2π i.e., 0 to 2π.

∴ t = 0 to t = π gives only length of half arc.

x = a(t-sin t) => $\frac{dx}{dt}$ = a (1-cos t)

y= a(1-cos t) => $\frac{dy}{dt}$ = a sin t

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\,[(1-\cos t)^2 + sin^2 t]$

$= 2a^2(1-\cos t)$

$= 2a^2 . 2sin^2\frac{t}{2}$

$= 4a^2 sin^2\frac{t}{2}$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2a \sin\frac{t}{2}$$

The required length = $\int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_0^{\pi} 2a\, sin\frac{1}{2}\, dt$$

$$= 2a \cdot \left[-\frac{\cos\left(\frac{t}{2}\right)}{1/2}\right]_0^{\pi}$$

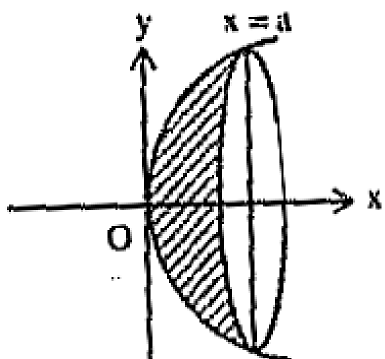
$$= -4a \left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$=4a$$

NOTE: This curve is known as cycloid.

22. Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$, bounded by its latus rectum about x-axis.

Solution:



The required surface area of the solid is obtained by revolving the area bounded by $y^2 = 4ax$, x = 0, x = a and x-axis about x-axis.

$\therefore$ s = $2\pi \int_0^{a} y\sqrt{1 + \left(\frac{dy}{dt}\right)^2}\, dx$

$$y^2 = 4ax$$

=> $2yy'$ = 4a

=> $y' = \frac{2a}{y}$

$1 + (y')^2 = 1 + \frac{4a^2}{y^2}$

$= \frac{y^2+4a^2}{y^2}$

$= \frac{4ax+4a^2}{y^2}$

$y \cdot \sqrt{1+(y')^2} = y \cdot \frac{\sqrt{4a} \cdot \sqrt{x+a}}{y}$

$= 2\sqrt{a} \cdot \sqrt{x+a}$

$S = 2\pi \int_0^a 2\sqrt{a}\,(x+a)^{1/2}\,dx$

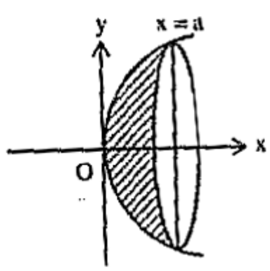
$= 4\sqrt{a}\pi \left[\frac{(x+a)^{3/2}}{3/2}\right]_0^a$

$= \frac{8\sqrt{a}\,\pi}{3} [(2a)^{3/2} - a^{3/2}]$

$= \frac{8a^2\pi}{3} [2\sqrt{2} - 1]$ sq.units.

23. Prove that the curved surface area of a sphere of radius *r* intercepted between two parallel planes at a distance *a* and *b* from the centre of the sphere is $2\pi r(b-a)$ and hence deduct the surface area of the sphere.($b > a$).

Solution:



The required surface area of the solid is obtained by revolving the area bounded by $x^2 + y^2 = r^2$, x = a, x = b, x-axis, about x-axis.

$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dt}\right)^2}\,dx$

$x^2 + y^2 = a^2$ => 2x + 2yy' = 0

=> $y' = -\frac{x}{y}$

$1 + (y')^2 = 1 + \frac{x^2}{y^2}$

$= \frac{x^2+y^2}{y^2}$

$= \frac{r^2}{y^2}$

$y \cdot \sqrt{1 + (y')^2} = y \cdot \frac{r}{y} = r$

$S = 2\pi \int_a^b r dx$

$= 2\pi r \; [x]_a^b$

S = 2πr (b - a) sq.units.

Deduction: To find the total surface area of the sphere , take the planes x = -r and x = r

i.e., instead of a and b take – r and r respectively.

s = 2πr (b - a) = 2πr [r – (- r)] = 4πr^2 sq.units

Six marks questions:

1.Evaulate $\int_0^1 \sqrt{9 - 4x^2}\, dx$

Solution:

$$\int_0^1 \sqrt{9 - 4x^2}\, dx = \int_0^1 2\sqrt{\left(\frac{3}{2}\right)^2 - x^2 dx}$$

$$= 2\left[\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{\left(\frac{3}{2}\right)^2}{2} sin^{-1}\left(\frac{x}{2/3}\right)\right]_0^1$$

$$= 2\left\{\left[\frac{1}{2}\sqrt{\left(\frac{3}{2}\right)^2 - 1} + \frac{9}{8} sin^{-1}\frac{3}{2}\right] - 0\right\}$$

$= 2\left[\frac{1}{2}\frac{\sqrt{5}}{2} + \frac{9}{8}sin^{-1}\frac{3}{2}\right] = \frac{\sqrt{5}}{2} + \frac{9}{4}sin^{-1}\frac{2}{3}$

2. Evaulate $\int_0^{\pi/4} sin^2x \ \ sin2x \ dx$

solution

Let sin x = t

Cos x = dt

x	0	$\pi/4$
t	0	$1/\sqrt{2}$

I = $4\int_0^{\frac{1}{\sqrt{2}}} t^3 \ dt \ \ = 4\left[\frac{t^4}{4}\right]_0^{\frac{1}{\sqrt{2}}}$

$= \frac{1}{4}$

3. Evaluate $\int_0^{\pi/2} \frac{\sin x \ dx}{9+cos^2x}$

Solution:

I = $\int_0^{\pi/2} \frac{\sin x \ dx}{9+cos^2x}$

let cos x = t

-sin x dx = dt

x	0	$\pi/2$
t	1	0

I = $\int_1^0 \frac{-dt}{3^2+t^2}$

$= -\frac{1}{3}\left[tan^{-1}\left(\frac{t}{3}\right)\right]_1^0$

$= -\frac{1}{3}\left\{tan^{-1}0 - tan^{-1}\left(\frac{1}{3}\right)\right\}$

$= \frac{1}{3}\tan^{-1}\left[\frac{1}{3}\right]$

4. Evaluate $\int_1^2 \frac{dx}{x^2+5x+6}$

Solution:

$$\int_1^2 \frac{dx}{x^2+5x+6} = \int_1^2 \frac{dx}{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)}\left[log\left(\frac{x+\frac{5}{2}-\frac{1}{2}}{x+\frac{5}{2}+\frac{1}{2}}\right)\right]_1^2$$

$$= \left[log\left(\frac{x+2}{x+3}\right)\right]_1^2$$

$$= \log\frac{4}{5} - \log\frac{3}{4}$$

$$= \log\left(\frac{4}{5} \times \frac{4}{3}\right)$$

$$= \log\left(\frac{16}{15}\right)$$

5. Evaluate $\int_0^1 \frac{(sin^{-1})^3}{\sqrt{1-x^2}}\ dx$

Solution:

$\int_0^1 \frac{(sin^{-1})^3}{\sqrt{1-x^2}}\ dx$

Let $\sin^{-1} x = t$

$\frac{1}{\sqrt{1-x^2}} dx = dt$

$I = \int_0^{\pi/2} r^3\ dt = \left[\frac{t^4}{4}\right]_0^{\pi/2}$

$= \frac{\pi^4}{64}$

6. Evaluate $\int_0^{\pi/2} sin2x \, cosx \, dx$

Solution:

I = $\int_0^{\frac{\pi}{2}} sin2x \, cosx \, dx \ = 2\int_0^{\frac{\pi}{2}} cos^2x \, . \sin x \, dx$

Let t = cos x

dt = -sinx dx

I = $2\int_1^0 -t^2 dt$

= $-2\left[\frac{t^3}{3}\right]_1^0$

$= \ -\frac{2}{3}[0-1]$

$= \frac{2}{3}$

7.Evaluate $\int_0^1 x^2 e^x \, dx$

Solution:

I = $\int_0^1 x^2 e^x \, dx$

Note : This problem can be done by using Bernoulli's formula instead of integration by parts.

i.e. $\int udv = uv - \ u^1 v_1 + u^{11} v_2 + u^{111} v_3 \dots \dots .$

u = x^2.u^1 = 2x .u^{11} = 2

dv = e^x, v = e^x , v_1 = e^x ; v_2 = e^x

I = $[x^2 e^x - 2xe^x + 2e^x]_0^1$

= $[e^x(x^2 - 2x + 2)]_0^1$

= [e(1-2+2)] – [1(2)]

= e-2

8. Evaluate : $\int_{-\pi/2}^{\pi/2} x\ sinx\ dx$

Solution:

Let f(x) = x sinx

f(-x) = (-x) sin(-x)

= x sinx

f(x) is a even function .

$\int_{-\pi/2}^{\pi/2} x\ sinx\ dx$ = 2 $\int_{0}^{\pi/2} x\ sinx\ dx$

= $2\left[\{x(-cosx)\} \ - \int_{0}^{\pi/2} (-\ cosx)\ dx\right]$

Using the method of integration by parts

= $2\left[\ 0\ +\ \int_{0}^{\pi/2} cosx\ dx\right]$

= $2[sin\ x]_{0}^{\pi/2}$

= 2[1-0]

= 2

9.Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\ dx$

Solution:

Let f(x) = sin^2x = $(sinx)^2$

f(x) = $(sin(-x))^2$

$= (-\sin x)^2$

$= \sin^2 x$

$= f(x)$

$\therefore f(x) \text{ is an even function}$

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = 2\int_0^{\pi/2} \sin^2 x \, dx$$

$= 2 \times \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$

$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \pi/2$

10. Evaluate $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)} dx$

Solution:

Let I = $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x)+f(\cos x)} dx$ ......(1)

$= \int_0^{\pi/2} \frac{f(\sin(\frac{\pi}{2}-x))}{f(\sin(\frac{\pi}{2}-x))+f(\cos(\frac{\pi}{2}-x))} dx$

$= \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x)+f(\sin x)} dx$ ......(2)

(1) + (2) => 2I = $\int_0^{\pi/2} \frac{f(\sin x) + f(\cos x)}{f(\sin x)+f(\cos x)} dx$

$= \int_0^{\pi/2} dx$

$= [x]_0^{\pi/2}$

$= \frac{\pi}{2}$

11. Evaluate $\int_0^1 x(1-x)^n$ dx

Solution:

Let I = $\int_0^1 x(1-x)^n$ dx

= $\int_0^1 (1-x)[1-(1-x)]^n$ dx $\quad\quad$ $\left[\int_0^a f(x)dx \;=\; \int_0^a f(a-x)dx\right]$

= $\int_0^1 (1-x)\, x^n \,dx$

= $\int_0^1 (\, x^n - \, x^{n+1})\, dx$

= $\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

= $\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

= $\frac{n+2-(n+1)}{(n+1)(n+2)}$

$\int_0^1 x(1-x)^n$ dx = $\frac{1}{(n+1)(n+2)}$

12.E valuate $\int_0^{\pi/2} \log(tanx)dx$

Solution:

Let I = $\int_0^{\pi/2} \log(tanx)dx$ ....(1)

= $\int_0^{\pi/2} log\left(\tan\left(\frac{\pi}{2} - x\right)\right)$ dx

= $\int_0^{\pi/2} \log(cotx)dx$ ....(2)

(1) +(2) =>2I = $\int_0^{\pi/2} [\log(tanx) + \log(cotx)]\, dx$

= $\int_0^{\pi/2} [\log(tanx)\, .\, \log(cotx)]\, dx$

$= \int_0^{\pi/2} (\log\ 1) dx$ (log 1= 0)

= 0

13. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{cotx}}$

Solution:

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot x}}$ ....(1)

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{cot\left(\frac{\pi}{2}-x\right)}}$ $\left(\int_a^b f(x)dx = \int_a^b f(a+b-x)\right)$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ ......(2)

Adding (1) and (2)

$2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{cosx}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right) dx$

$= \int_{\pi/6}^{\pi/3} dx$

$=[x]_{\pi/6}^{\pi/3}$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$I = \frac{\pi}{12}$

14. Evaluate $\int_0^{\pi/2} sin^3 \cos x \, dx$

Solution:

I = $\int_0^{\pi/2} sin^3 \cos x \, dx$ .....(1)

Let f(x) = $\sin^3 x . \cos x$ f(a-x) = $f\left(\frac{\pi}{2} - x\right)$

= $\cos^3 x . \sin x$

Again I = $\int_0^{\pi/2} cos^3 x . \sin x \, dx$ ......(2)

Adding (1) and (2)

2I = $\int_0^{\frac{\pi}{2}} \sin x \cos x (sin^2 x + cos^2 x) \, dx$

= $\frac{1}{2} \int_0^{\pi/2} \sin 2x \, dx = \frac{1}{2} \left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$

= $-\frac{1}{4} [cos\pi - \cos 0]$

2I = $-\frac{1}{4} [-1 - 1] = \frac{1}{2}$

∴ I = $\frac{1}{4}$

15. Evaluate $\int_{-\pi/2}^{\pi/2} cos^3 x \, dx$

Solution:

I = $\int_{-\pi/2}^{\pi/2} cos^3 x \, dx$

Let f(x) = $[\cos(-x)]^3 = \cos^3 x$ = f(x)

∴ f(x) is even function.

$$\therefore I = 2\int_0^{\pi/2}\left[\frac{3\cos x + \cos 3\,x}{4}\right]dx$$

$$= \frac{1}{2}\left[3\sin x + \frac{\sin 3x}{3}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(3 - \frac{1}{3}\right) - 0\right]$$

$$= \frac{4}{3}$$

16. Evaluate $\int_{-\pi/2}^{\pi/2} sin^2 x \cos x \, dx$

Solution:

I = $\int_{-\pi/2}^{\pi/2} sin^2 x \cos x \, dx$

Let f(x) = $sin^2 x \cos x$

f(-x) = $[\sin(-x)]^2[\cos(-x)]$

= $\sin^2$ x .cosx

= f(x)

∴ f(x) is even function

∴ I = $2\int_0^{\pi/2} sin^2 x \cos x \, dx$

$$= 2\left[\frac{sin^3 x}{3}\right]_0^{\frac{\pi}{2}}$$

$= \frac{2}{3}[1-0]$

$= \frac{2}{3}$

17. Evaluate $\int_0^1 log\left(\frac{1}{x}-1\right)dx$

Solution:

Let I $= \int_0^1 log\left(\frac{1}{x}-1\right)dx$ ......(1)

Since $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

$\therefore I = \int_0^1 log\left(\frac{1}{1-x}-1\right)$dx ......(2)

Adding (1) and (2)

2I = $\int_0^1 log\left(\frac{1}{x}-x\right)\times\left(\frac{1}{1-x}-x\right)$ dx

$= \int_0^1 log\left(\frac{1-x}{x}\times\frac{x}{1-x}\right)$dx

$= \int_0^1 log\,1\,dx$

$= \int_0^1 0\,dx = 0 \;\rightarrow\; I = 0$

18. Evaluate $\int_0^3 \frac{\sqrt{x}dx}{\sqrt{x}+\sqrt{3-x}}$

Solution:

I = $\int_0^3 \frac{\sqrt{x}\ dx}{\sqrt{x}+\sqrt{3-x}}$ .....(1)

I = $\int_0^3 \frac{\sqrt{3-x}}{\sqrt{3-x}.\sqrt{3-(3-x)}}dx$

= $\int_0^3 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}$dx ....(2)

(1)+(2)=>

2I = $\int_0^3 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}$dx

= $\int_0^3 dx$

= $[x]^3{}_0$

=3

I = $\frac{3}{2}$

19.Evaluate$\int_0^1 x(1-x)^{10}dx$

Solution:

I = $\int_0^1 x(1-x)^{10}dx$

= $\int_0^1 (1-x)[1-(1-x)]^{10}\ dx$

$= \int_0^1 (1-x)x^{10}\,dx$

$= \int_0^1 (x^{10} - x^{11})\,dx$

$= \left[\frac{x^{11}}{11} - \frac{x^{12}}{12}\right]_0^1$

$= \frac{1}{11} - \frac{1}{12}$

$= \frac{1}{132}$

20. Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$

Solution:

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$ ......(1)

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{tan\left(\frac{\pi}{2}-x\right)}}$ $\left(\int_a^b f(x)dx = \int_a^b f(a+b-x)\right)$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ ......(2)

Adding (1) and (2)

$2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{sinx}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right) dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$2I = \frac{\pi}{6}$

$I = \frac{\pi}{12}$

21.Evaluate $\int sin^5 x\, dx$

Solution:

$I_n = \int sin^5 x\, dx$

$I_n = -\frac{1}{n}[\sin^{n-1}x.\cos x] + \frac{n-1}{n} I_{n-2}$ .....(I)

$\therefore \int sin^5 x\, dx = I_5$

$= -\frac{1}{5}[\sin^4 x.\cos x] + \frac{4}{5} I_3$ (when n= 5 in I)

$= -\frac{1}{5}[\sin^4 x.\cos x] + \frac{4}{5}[-\frac{1}{3}\sin^2 x.\cos x] + \frac{2}{3} I_1$ (when n=3 in I)

$\int sin^5 x\, dx = -\frac{1}{5}\sin^4 x.\cos x - \frac{4}{15}\sin^2 x.\cos x + \frac{8}{15} I_1$ ......(II)

$I_1 = \int sin^1 x\, dx = -\cos x + c$

$\therefore \int sin^5 x\, dx = -\frac{1}{5}\sin^4 x.\cos x - \frac{4}{15}\sin^2 x.\cos x - \frac{8}{15}\cos x + c$

22. Evaluate $\int sin^6 x\, dx$

Solution:

$I_n = \int sin^6 x\, dx$

$I_n = -\frac{1}{n}[\sin^{n-1}x.\cos x] + \frac{n-1}{n} I_{n-2}$ .....(I)

$\therefore \int sin^6 x\, dx = I_6$

$I_6 = -\frac{1}{6}[\sin^5 x.\cos x] + \frac{5}{6} I_4$ (when n = 6 in I)

$= -\frac{1}{6}\sin^5 x.\cos x + \frac{5}{6}[-\frac{1}{4}\sin^3 x.\cos x + \frac{3}{4} I_2]$ (when n=4 in I)

$= -\frac{1}{6}\sin^5 x.\cos x - \frac{5}{24}\sin^3 x.\cos x + \frac{5}{8} I_2$ (when n=2 in I)

$= -\frac{1}{6}\sin^5 x.\cos x - \frac{5}{24}\sin^3 x.\cos x + \frac{5}{8}[-\frac{1}{2}\sin x.\cos x + \frac{1}{2}\ I_0$

$I_0 = \int \sin^0 x\, dx = x + c$

$\int \sin^6 x\, dx = -\frac{1}{6}\sin^5 x.\cos x - \frac{5}{24}\sin^3 x.\cos x\ \frac{5}{16}\sin x.\cos x + \frac{5}{16}\ x + c$

23. Evaluate $\int_0^{2\pi} \sin^9 \frac{x}{4}\, dx$

Solution:

Put $t = \frac{x}{4}$

dx = 4 dt

x	0	2π
t	0	$\pi/2$

$\int_0^{2\pi} \sin^9 \frac{x}{4}\, dx = 4\int_0^{2\pi} \sin^9 t\, dt$

$= 4\left(\frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1\right)$

$= \frac{512}{315}$

24. Evaluate $\int_0^{\pi/6} cos^7 3x$ dx

Solution:

Put t = $3x$

dx = $\frac{1}{3}$ dt

x	0	$\pi/6$
t	0	$\pi/2$

$\int_0^{\pi/6} cos^7 3x$ dx = $\frac{1}{3}\int_0^{\pi/2} cos^7 t$ dt

$= \frac{1}{3}\left(\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1\right)$

$= \frac{16}{105}$

25. Evaluate $\int_0^{\pi/2} sin^4 x \;\; cos^2 x$ dx

Solution:

$\int_0^{\pi/2} sin^4 x \;\; cos^2 x$ dx = $\int_0^{\pi/2} sin^4 x\,(1 - \; sin^2 x)$ dx

$= \int_0^{\pi/2}(sin^4 x \; - \; sin^6 x)$ dx

$=\int_0^{\pi/2} sin^4 x\, dx \; - \int_0^{\pi/2} s\,in^6 x)$ dx

$= \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} - \frac{5}{4}.\frac{3}{2}.\frac{1}{2}.\frac{\pi}{2}$

$= \frac{\pi}{32}$

26. Evaluate $\int x^3 e^{2x} dx$

Solution:

Using Bernoulli's formula

$$\int u\,dv = uv - u' v_1 + u'' v_2 \ldots$$

dv = e^{2x} dx

u = x^3	v = $\frac{e^{2x}}{2}$
$u' = 3x^2$	$v_1 = \frac{e^{2x}}{4}$
$u'' = 6x$	$v_2 = \frac{e^{2x}}{8}$
$u''' = 6$	$v_3 = \frac{e^{2x}}{16}$

$$\int x^3 e^{2x} dx = (x^3)\left(\frac{e^{2x}}{2}\right) - (3x^2)\left(\frac{e^{2x}}{4}\right) + (6x)\left(\frac{e^{2x}}{8}\right) - (6)\left(\frac{e^{2x}}{16}\right)$$

$$= \frac{e^{2x}}{2}\left[x^3 - \frac{3}{2}x^2 + \frac{3x}{2} - \frac{3}{4}\right]$$

27.Evaluate $\int_0^1 x\, e^{-4x}$ dx

Solution:

Using Bernoulli's formula

$$\int \mathrm{u\, dv} = \mathrm{uv} - \mathrm{u'v_1} + \mathrm{u''\, v_2} \ldots$$

$$dv = e^{-4x}\ dx$$

u = x $\qquad$ v = - $\frac{e^{-4x}}{4}$

u' = 1 $\qquad$ $v_1 = \frac{e^{-4x}}{16}$

$$\int_0^1 x\, e^{-4x}\ dx = \left[(x)\left(-\ \frac{e^{-4x}}{4}\right) - (1)\left(\frac{e^{-4x}}{16}\right)\right]_0^1$$

$$= \left(-\ \frac{e^{-4x}}{4} - 0\right) - \frac{1}{16}(e^{-4} - e^{0})$$

$$= \frac{1}{16} - \frac{5}{16}e^{-4}$$

28. Evaluate $\int sin^4 x$ dx

Solution:

$\int sin^4 x$ dx = I_4

$I_n = -\frac{1}{n}[\sin^{n-1}x.\cos x] + \frac{n-1}{n} I_{n-2}$

$I_4 = -\frac{1}{4}[\sin^3 x.\cos x] + \frac{3}{4} I_2$

$I_2 = -\frac{1}{2}[\sin x.\cos x] + \frac{1}{2} I_0$

$I_0 = \int dx = x$

$I_4 = -\frac{1}{4}[\sin^3 x.\cos x] + \frac{3}{4}\left\{-\frac{1}{2}[\sin x.\cos x\,] + \frac{1}{2}x\right\}$

$= -\frac{1}{4}\sin^3 x.\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + c$

29. Evaluate $\int cos^5 x dx$

Solution:

$\int cos^5 x dx = I_5$

$I_n = \frac{1}{n}[\cos^{n-1} x.\sin x] + \frac{n-1}{n} I_{n-2}$

$I_5 = \frac{1}{5}\cos^4 x.\sin x + \frac{4}{5} I_3$

$I_3 = \frac{1}{3}\cos^2 x.\sin x + \frac{2}{3} I_1$

$I_1 = \int \cos dx = \sin x$

$I_5 = \frac{1}{5}\cos^4 x.\sin x + \frac{4}{5}\left(\frac{1}{3}cos^2 x.sinx + \frac{2}{3}sinx\right)$

$= \frac{1}{5}\cos^4 x.\sin x + \frac{4}{15}\cos^2 x.\sin x + \frac{8}{15}\sin x + c$

30. Evaluate $\int_0^{\pi/2} sin^6 x\, dx$

Solution:

$\int_0^{\pi/2} sin^6 x\, dx = I_6$

$I_n = \frac{n-1}{n}.\frac{n-3}{n-2} \ldots\ldots\ldots \frac{1}{2}\frac{\pi}{2}$ if n is even

$I_6 = \frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{5\pi}{32}$

31. Evaluate $\int_0^{\pi/2} cos^9 x\, dx$

Solution:

$\int_0^{\pi/2} cos^9 x\, dx = I_9$

$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2}$ .... $\frac{2}{3}$ I if n is odd

$I_9 = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I = \frac{128}{315}$

32.Evaluate $\int_0^{\pi/4} cos^8 2x\, dx$

Solution:

Put t = 2x

dt = 2 dx => dx = $\frac{dt}{2}$

x	0	$\pi/4$
t	0	$\pi/2$

$\int_0^{\pi/4} cos^8 2x\, dx = \frac{1}{2} \int_0^{\pi/2} cos^8 t\, dt$

$= \frac{1}{2} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{512}$

33. Evaluate $\int_0^{\pi/6} sin^7 3x\,dx$

Solution:

Put t = 3x

dt = 3 dx =.> dx = $\frac{dt}{3}$

x	0	$\pi/6$
t	0	$\pi/2$

$\int_0^{\pi/6} sin^7 3x\,dx = \frac{1}{3}\int_0^{\pi/6} sin^7 t\,dt$

$= \frac{1}{3}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}$

$= \frac{16}{105}$

34.Evaluate $\int_0^1 x\,e^{-2x}$ dx

Solution:

Using Bernoulli's formula

$$\int \mathrm{u\,dv = uv - u'v_1 + u''\,v_2\,...}$$

dv = e^{-2x} dx

u = x v = - $\frac{e^{-2x}}{2}$

u' = 1 $v_1 = \frac{e^{-2x}}{4}$

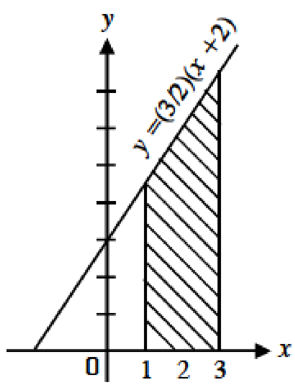
$\int_0^1 x\, e^{-2x}$ dx = $\left[(x)\left(-\ \frac{e^{-2x}}{2}\right) - \ (1)\left(\frac{e^{-2x}}{4}\right)\right]_0^1$

$= \left(-\ \frac{e^{-2}}{2} - \ 0\right) - \ \frac{1}{4}(e^{-2} - e^0)$

$= \frac{1}{4} - \ \frac{3}{4}e^{-2}$

35.Find the area of the region bounded by the line $3x - 2y + 6 = 0, x = 1, x = 3$ and x-axis.

Solution:



Since the line $3x - 2y + 6 = 0$ lies above the x-axis in the interval [1, 3],

(i.e., $y > 0$ for $x \in (1,3)$)

Required area = $\int_1^3 y\, dx = \frac{3}{2}\int_1^3 (x+2)\, dx$

$= \frac{3}{2}\left[\frac{x^2}{2} + 2x\right]_1^3$

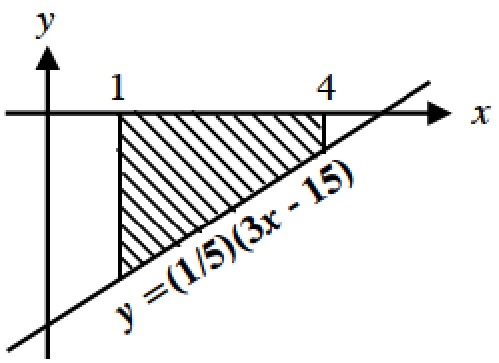
$= \frac{3}{2}\left[\frac{1}{2}(9-1) + \ 2(3-1)\right]$

$= \frac{3}{2}(4+4)$

$= 12$ sq. units

36. Find the area of the region bounded by the line $3x - 5y - 15 = 0$, $x = 1$, $x = 4$ and x-axis.

Solution:



The line $3x - 5y - 15 = 0$ lies below the x-axis in the interval $x = 1$ and $x = 4$

$\therefore$ Required area $= \int_1^4 (-y)\, dx$

$= \int_1^4 -\frac{1}{5}(3x - 15)\, dx$

$= \frac{3}{5}\int_1^4 (5 - x)\, dx$

$= \frac{3}{5}\left[5x - \frac{x^2}{2}\right]_1^4$

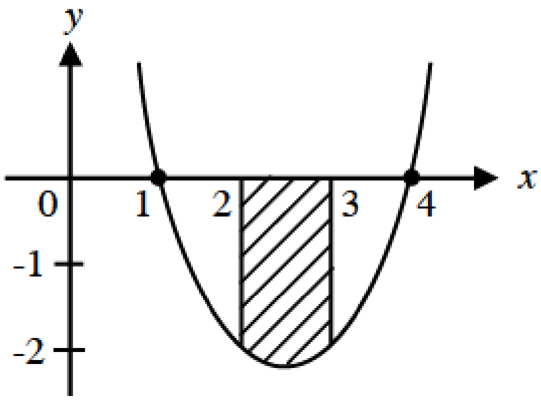
$= \frac{3}{5}\left[5(4-1) - \frac{1}{2}(16-1)\right]$

$= \frac{3}{5}\left[15 - \frac{15}{2}\right]$

$= \frac{9}{2} sq.\ units.$

37. Find the area of the region bounded $y = x^2 - 5x + 4$, $x = 2$, $x = 3$ and the x-axis.

Solution:



For all x, $2 \leq x \leq 3$ the curve lies below the x-axis.

Required area $= \int_2^3 (-y)\, dx$

$$= \int_2^3 -(x^2 - 5x + 4)\, dx$$

$$= \left[\frac{x^3}{3} - 5\frac{x^2}{2} + 4x\right]_2^3$$

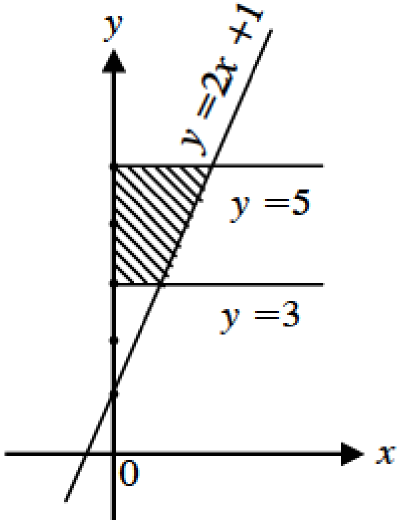
$$= -\left[\left(9 - \frac{45}{2} + 12\right) - \left(\frac{8}{3} - \frac{20}{2} + 8\right)\right]$$

$$= -\left[\frac{-13}{6}\right]$$

$= \frac{13}{6}$ sq. units

38. Find the area of the region bounded by $y = 2x + 1$, $y = 3$, $y = 5$ and y – axis.

Solution :



The line $y = 2x + 1$ lies to the right of y-axis between the lines

$y = 3$ and $y = 5$.

$\therefore$ The required area $A = \int_c^d x\, dy$

$$= \int_3^5 \left(\frac{y-1}{2}\right) dy$$

$$= \frac{1}{2}\int_3^5 (y-1)dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - y\right]_3^5$$

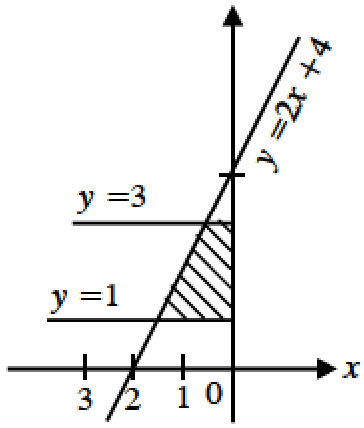
$$= \frac{1}{2}\left[\left(\frac{25}{2} - \frac{9}{2}\right) - (5-3)\right]$$

$$= \frac{1}{2}\ [8\text{-}2]$$

= 3 sq.nuits

39. Find the area of the region bounded $y = 2x + 4$, $y = 1$ and $y = 3$ and y-axis

Solution:



The curve lies to the left of y-axis between the lines $y = 1$ and $y = 3$

$\therefore$ Required area $A = \int_1^3 (-x)\, dy$

$$= \int_1^3 -\left(\frac{y-4}{2}\right) dy$$

$$= \frac{1}{2}\int_1^3 (4-y)\, dy$$

$$= \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_1^3$$

$$= \frac{1}{2}\,[8\text{-}4]$$

= 2 sq.units

40.(i) Evaluate the integral $\int_1^5 (x-3)\, dx$

(i) Find the area of the region bounded by the line $y + 3 = x$, $x = 1$ and $x = 5$

Solution :

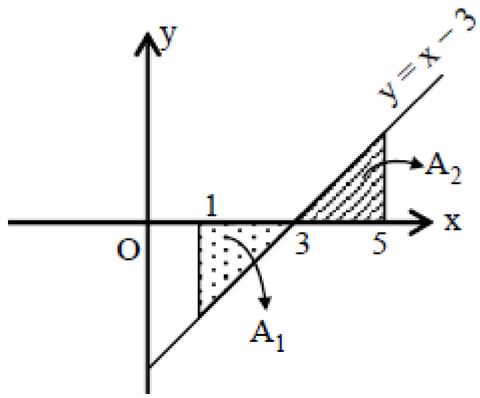
(i) $\int_1^5 (x-3)\, dx = \left[\frac{x^2}{2} - 3x\right]_1^5$

$$= \left(\frac{25}{2} - 15\right) - \left(\frac{1}{2} - 3\right)$$

$$= 12 - 12 = 0 \quad \text{…….(I)}$$

(ii) The line $y = x - 3$ crosses x-axis at $x = 3$

From the diagram it is clear that $A1$ lies below x-axis.



$\therefore A1 = \int_1^3 (-y)dx$

As $A2$ lies above the x-axis

$A_2 = \int_3^5 y\, dx$

$\therefore \text{Total area} = \int_1^5 (x-3)dx$

$= \int_1^3 -(x-3)\, dx + \int_3^5 (x-3)\, dx$

= (6 - 4) + (8 – 6)

= 2 +2

= 4 sq.units ….(II)

Note:

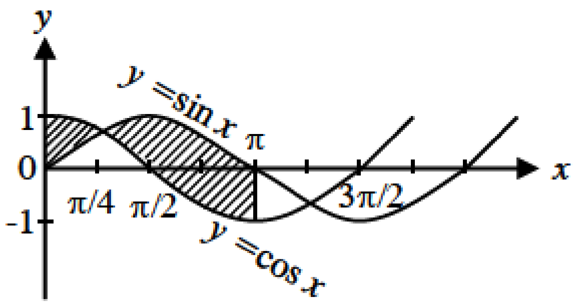
From I and II it is clear that the integral f(x) is not always imply an area. The fundamental theorem asserts the anti-derivative method works even when the function f(x) is not always positive

41. Compute the area between the curve $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

Solution : To find the points of intersection solve the two equations.

$\text{Sin } x = \cos x = \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\sin x = \cos x = -\frac{1}{\sqrt{2}} = \frac{5\pi}{4}$



From the figure we see that $\cos x > \sin x$ for $0 \le x < \frac{\pi}{4}$ and $\sin x > \cos x$ for $\frac{\pi}{4} < x < \pi$

Area A $= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$

$= (\sin x + \cos x)_0^{\frac{\pi}{4}} + (\sin x + \cos x)_{\pi/4}^{\pi}$

$= (\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) - (\sin 0 - \cos 0) + (-\cos\pi - \sin\pi) - (-\cos\frac{\pi}{4} - \sin\frac{\pi}{4})$

$= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (0+1) + (1-0) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}})$

$= 2\sqrt{2}$ sq.units

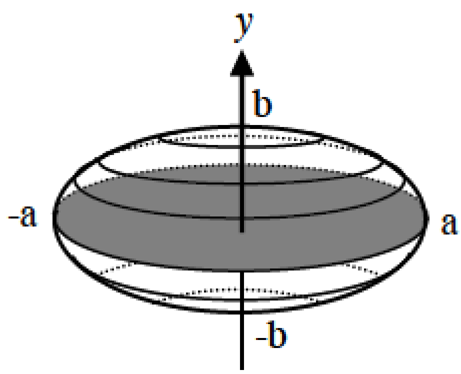
42. Find the volume of the solid that results when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b > 0$) is revolved about the minor axis.

Solution :

Volume of the solid is obtained by revolving the right side of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y-axis.

Limits for y is obtained by putting

$x = 0 \Rightarrow y^2 = b^2 \Rightarrow y = \pm b$



From the given curve $x^2 = \frac{a^2}{b^2}(b^2 - y^2)$

$\therefore$ Volume is given by

$$V = \int_c^d \pi y^2 \, dy$$

$$= \int_{-b}^{b} \pi \frac{a^2}{b^2}(b^2 - y^2) \, dy$$

$$= 2\pi \frac{a^2}{b^2}\left(b^2 y - \frac{y^3}{3}\right)_0^b$$

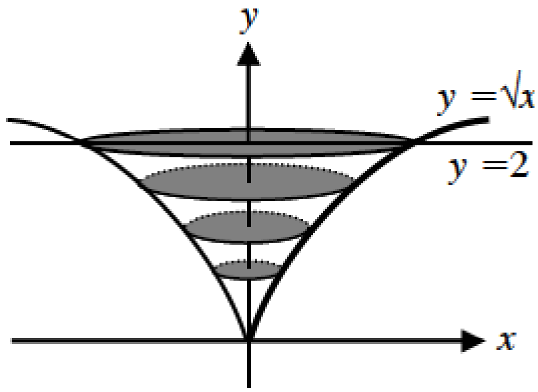
$$= 2\pi \frac{a^2}{b^2}\left(b^3 - \frac{b^3}{3}\right)$$

$$= \frac{4\pi}{3} a^2 b \text{ cubic units}$$

43.Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the y-axis.

Solution :

Since the solid is generated by revolving about the y-axis, rewrite $y = x$ as $x = y^2$.



Taking the limits for y, $y = 0$ and $y = 2$
(putting $x = 0$ in $x = y^2$, we get $y = 0$)

Volume is given by V = $\int_c^d \pi y^2 \, dy$

$= \int_0^2 \pi y^4 \, dy$

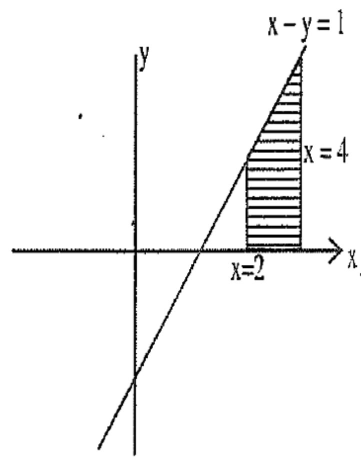
$= \left[\frac{\pi y^5}{5}\right]_0^2$

$= \frac{32\pi}{5}$ cubic units.

44. Find the area of the region bounded by the line $x - y = 1$ and (i) x-axis, $x = 2$ and $x = 4$ (ii) x-axis, $x = -2$ and $x = 0$

Solution:

The required area is bounded by the lines
$x - y = 1 \Rightarrow y = x - 1$, $x = 2$, $x = 4$ and x- axis
The area lies above x- axis



Area = $\int_a^b y \, dx$

$= \int_2^4 (x - 1) \, dx$

$= \left[\frac{x^2}{2} - x\right]_2^4$

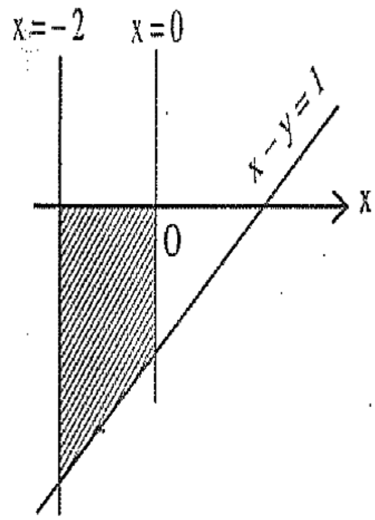
$= \left(\frac{16}{2} - 4\right) - \left(\frac{4}{2} - 2\right)$

= 4 sq. units

(ii)The required area is bounded by the lines x –y =1 , x = -2, x=0 and x-axis. Here the area lies below x-axis

x –y =1 => y = x - 1

Area = $\int_{-2}^{0}(-y\,)dx = \int_{0}^{-2} y\, dx$



$= \int_{0}^{-2} (x - 1)\, dx$

$= \left[\frac{x^2}{2} - x\right]_0^{-2}$

= ($\frac{4}{2}$ + 2) - (0 + 0)

= 4 sq.units

45. Find the area of the region bounded by the line $x - 2y - 12 = 0$ and

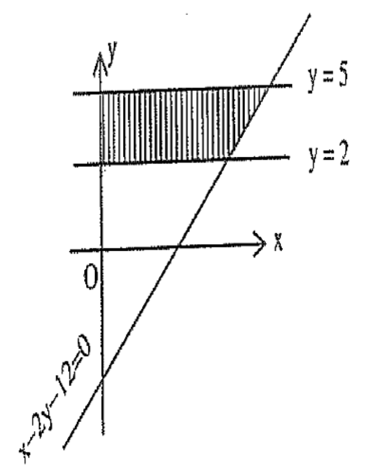
(i) y-axis, $y = 2$ and $y = 5$ (ii) y-axis, $y = -1$ and $y = -3$

Solution:

(i)the required area is bounded by the lines

$x - 2y - 12 = 0$ => x = 2y -12 , y = 2, y= 5 and x-axis

The area lies right of y-axis



Area = $\int_{c}^{d} x\, dy$

$= \int_{2}^{5}(2y + 12)\, dy$

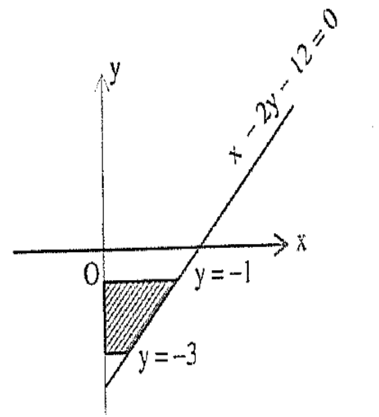
$=\left[2\frac{y^2}{2} + 12y\right]_2^5$

= (25+ 60) – (4 + 24)

= 57 sq. units

(ii)The required area I bounded by the lines x-2y 12 =0, y= -3 , y = - 1 and y-axis. This area lies right of y-axis

Area = $\int_{-1}^{-3}(-x\,)dy \quad = \quad \int_{-3}^{-1}(-x\,)dy$



$= \int_{-3}^{-1}(2y + \ 12)\ dy$

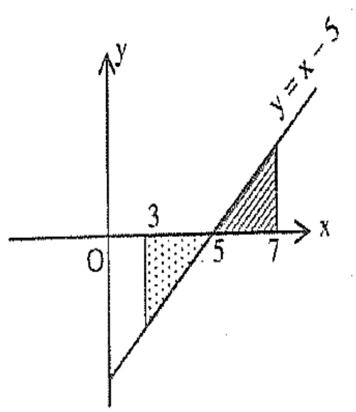
$= \left[2\frac{y^2}{2} + \ 12y\right]_{-3}^{-1}$

= (1 – 12) – (9 – 36)

= 16 sq.units

46. Find the area of the region bounded by the line $y = x - 5$ and the x-axis between the ordinates $x = 3$ and $x = 7$.

Solution:



The required area lies partially above x-axis and partially below x-axis. Therefore split the area into two pieces.

One piece is bounded by the line y = x – 5 , x = 3, x = 5 and x-axis. But this area lies below x-axis.

$$\therefore \text{Area of the part is } \int_3^5 (-y)\ dx$$

Other piece is bounded by the line y = x – 5 , x = 5, x = 7 and x-axis,. But this area lies above x-axis.

$$\therefore \text{Area of this part is } \int_5^7 y\ dx$$

Sum of this two parts = $\int_3^5 (-x+5)\,dx + \int_5^7 (x-5)\,dx$

$$=\left[-\frac{x^2}{2} + 5x\right]_3^5 + \left[\frac{x^2}{2} - 5x\right]_5^7$$

$$= \left[\left(\frac{-25}{2} + 25\right) - \left(\frac{-9}{2} + 15\right)\right] + \left[\left(\frac{49}{2} - 35\right) - \left(\frac{25}{2} - 25\right)\right]$$

= 4 sq.units

47. Find the area of the region bounded by $x^2 = 36y$, y-axis, $y = 2$ and $y = 4$.

Solution:

The required area is bounded by the curve $x = 6\sqrt{y}$, the lines y = 2, y = 4 and y-axis. It lies right of y-axis.

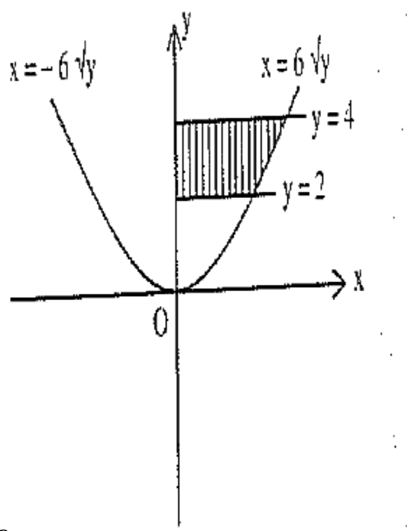
Area A = $\int_2^4 x\,dy$

$= \int_2^4 6\sqrt{y}\,dy$

$= 6\left[\frac{y^{3/2}}{3/2}\right]_2^4$

$= 6 \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_2^4$

$= 6[4 - \sqrt{2}]$ sq.units



48. Find the area included between the parabola $y^2 = 4ax$ and its latus rectum.

Solution:

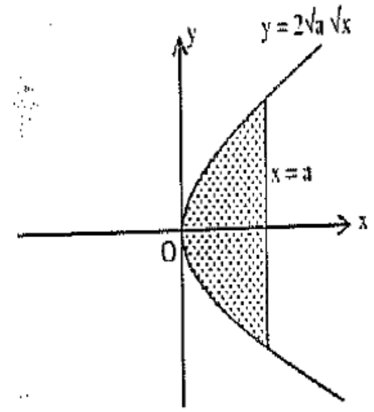
The area is twice the area bounded by the curve $y = 2\sqrt{a}\sqrt{x}$, x = 0, x = a and x-axis. [Since it is symmetrical about x-axis].

Required area = $2\int_0^a y\,dx$

$= 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx$

$= 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a$

$= \frac{8\sqrt{a}}{3}\left[a^{3/2} - 0\right]$



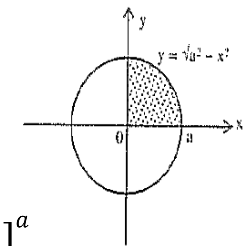
$= \frac{8a^2}{3}$ sq. units

49 . Find the area of the circle whose radius is a

Solution:

Take the centre of the circle as (0,0).the equation of the circle is $x^2 + y^2 = a^2$

Since it is symmetrical about both axes, the required area is 4 times the area in the first quadrant. The first quadrant area is bounded by the curve



y = $\sqrt{a^2 - x^2}$, x = 0 x = a and x-axis.

$\therefore$ the required area = $4\int_0^a y dx$ = $4\int_0^a \sqrt{a^2 - x^2} dx$

$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= 4\left\{\left(0 + \frac{a^2}{2} sin^{-1} 1\right) - (0)\right\}$

$= 4\left(\frac{a^2}{2} . \frac{\pi}{2}\right) = \pi a^2$ sq. units

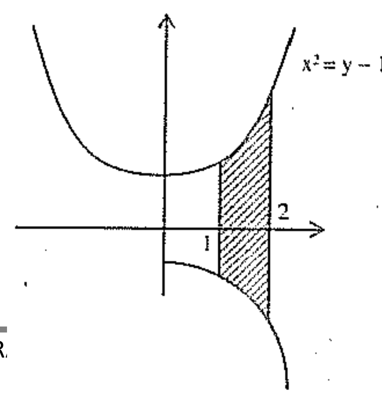
50. Find the volume of the solid that results when the region enclosed by the

given curves:

(i) . y = 1+x^2 , x = 1, x = 2 , y = 0 is revolved about the x-axis.

Solution:

The volume is obtained by revolving the area bounded by y = 1+x^2 , x = 1, x = 2 , x-axis about x-axis.



Required volume = $\pi\int_1^2 y^2 dx$

$= \pi\int_1^2 (1 + x^2)^2 dx$

$= \pi\int_1^2 (x^4 + 2x^2 + 1) dx$

$$= \pi\left[\frac{x^2}{5} + \frac{2x^3}{3} + x\right]_1^2$$

$$= \pi\left[\left(\frac{32}{5} + \frac{16}{3} + 2\right) - \left(\frac{1}{5} + \frac{2}{3} + 1\right)\right]$$

$$= \pi\left\{\frac{178}{15}\right\}$$

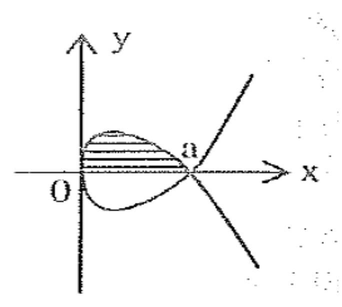
$$= \frac{178\pi}{15} \quad \text{cubic units.}$$

(ii) $2ay^2 = x(x-a)^2$ is revolved about x-axis , a>0.

Solution:

The curve is symmetric about x-axis and it passes through the origin. It cuts the x-axis at x = 0, x = a(twice). Clearly a loop is formed between x = 0 and x = a. as x→ ∞ , y → ±∞ .The required volume is obtained by , revolving the are bounded by the curve.

$2ay^2 = x(x-a)^2, x = 0$ and x-axis, about x-axis



∴The required volume = $\pi\int_0^a y^2\, dx$

$$= \pi\int_0^a \frac{1}{2a}\, x(x-a)^2\, dx$$

$$= \frac{\pi}{2a}\left[\frac{x^4}{4} - 2a\,\frac{x^3}{3} + a^2\,\frac{x^2}{2}\right]_0^a$$

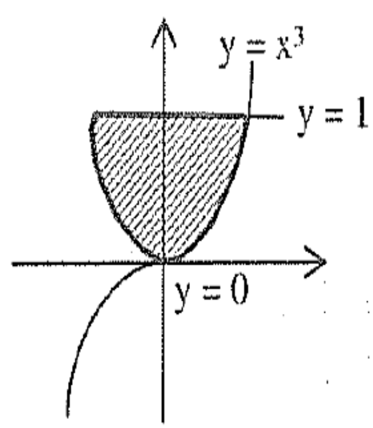
$$= \frac{\pi}{2a}\left[\left(\frac{a^4}{4} - \frac{2a^4}{3} + \frac{a^4}{2}\right) - 0\right]$$

$$= \frac{\pi a^3}{24} \text{ cubic units.}$$

(iii). y = x^3, x = 0, y = 1 is revolved about the y-axis.

Solution:

The required volume = $\pi\int_0^1 x^2\, dy$



$$= \pi\int_0^1 y^{2/3}\, dy$$

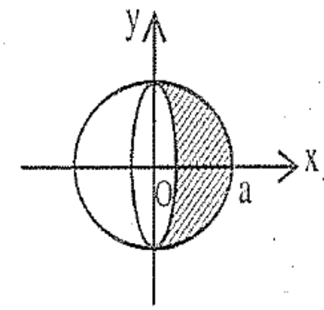
$$= \pi\left[\frac{y^{5/3}}{5/3}\right]_0^1$$

$= \frac{3\pi}{5}$ cubic units.

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about major axis a>b>0.

Solution:

The required volume is twice the volume obtained by revolving the area in the first quadrant about x-axis .



The first quadrant area is bounded by the curve.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; x = 0, x = a and x-axis.

Since the area is revolving about x-axis (major axis),

The required volume = $2\pi\int_0^a y^2\, dx$

$$= 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)dx$$

$$= 2\pi\frac{b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi\frac{b^2}{a^2}\left[\left(a^2x - \frac{x^3}{3}\right) - 0\right]$$

$$= \frac{4}{3}\pi\, ab^2 \text{ cub.units}$$

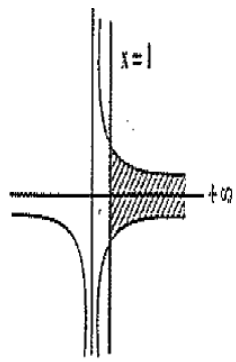
i.e.,The volume obtained by revolving the ellipse about its major axis is

$\frac{4}{3}\pi\, ab^2$ cubic units

(v) The area of the region bounded by the curve xy = 1, x- axis, x = 1 and x = ∞ find the volume of the solid generated by revolving the area mentioned about x-axis.

Solution:

The required volume is obtained by revolving the area bounded by the curve xy = 1. X = ± ∞ and x-axis, about x-axis.



∴ Required volume = $\pi\int_{1}^{\infty} y^2\, dx$

$= \pi\int_{1}^{\infty} \frac{1}{x^2}\, dx$

$= \pi\left[-\frac{1}{x}\right]_{1}^{\infty}$

$= -\pi\left[\frac{1}{\pi} - 1\right]$

$= -\pi\,[0 - 1]$

$= \pi$ cubic units.

DIFFERENTIAL EQUATIONS

Six marks questions

1. Form the differential equation by eliminating arbitrary constants given in brackets against y= Ae^{2x} cos(3x+B) {A,B}

Solution:

e^{-2x} y= A cos(3x+B)

Differentiating twice we have

$y' e^{-2x}$-2 e^{-2x} y = -3A sin(3x+B)

$-2\ e^{-2x}y' + e^{-2x}y'' + 4e^{-2x}y - 2\ e^{-2x}y' = -9A\cos(3x + B)$

$e^{-2x}(y'' + 4y - 4y') = -9\ A\cos$(3x+B)

$y'' + 4y - 4y' = -9$[Ae^{2x} cos(3x+B)]

$y'' + 4y - 4y' = -9y$

$y'' - 4y' + 13y = 0$

2. Find the differential equation of the family of straight lines y=mx +$\frac{a}{m}$ when

(i) m is a parameters; (ii) a is a parameter; (iii) a, m is a parameters

Solution:

(i) m is a parameter

$$\frac{dy}{dx} = m$$

$y' = m$

$y = y'x + \frac{a}{y'}$

$y = (y')^2x+a$

(ii) a is a parameter

$y' = m$

(iii) a, m both are parameters

$y' = m$

$y'' = 0$

3. Find the differential equation that will represent the family of all circles having centres on the x-axis and the radius is unity.

Solution :

$(x-a)^2 + y^2 = 1$ ....(1)

$(x-a) + y\, y' = 0$

$(x-a) = -y\, y'$

$(1) \Rightarrow y^2 y'^2 + y^2 = 1$

$y^2[(y')^2 + 1] = 1$

4. Form the differential equation form the equation is $A x^2 + B y^2 = 1$

Solution :

$Ax^2 + By^2 = 1 \cdots (1)$

Differentiating, $2Ax + 2Byy' = 0$ i.e., $Ax + Byy' = 0 \cdots(2)$

Differentiating again, $A + B(yy'' + y'^2) = 0 \cdots(3)$

Eliminating A and B between (1), (2) and (3) we get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy' & 0 \\ 1 & yy'' + y'^2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (yy'' + y'^2)\,x - yy' = 0$$

5. Solve : $\frac{dy}{dx} = 1 + x + y + xy$

Solution : The given equation can be written in the form

$$\frac{dy}{dx} = (1 + x) + y(1 + x)$$

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

$$\frac{dy}{1+y} = (1 + x)dx$$

Integrating, we have

$\log(1 + y) = x + \frac{x^2}{2} + c$, which is the required solution.

6. Solve $3e^x \tan y\, dx + (1 + e^x)\sec^2 y\, dy = 0$

Solution :

The given equation can be written in the form

$$\frac{3e^x}{1+e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we have

$3 \log (1 + e^x) + \log \tan y = \log c$

$\Rightarrow \log [\tan y (1 + e^x)^3] = \log c$

$\Rightarrow (1 + e^x)^3 \tan y = c$, which is the required solution

7. Solve : $\frac{dy}{dx} + (\frac{1-y^2}{1-x^2})^{1/2} = 0$

Solution :

The given equation can be written as

$\frac{dy}{dx} = -(\frac{1-y^2}{1-x^2})^{1/2}$

$\frac{dy}{\sqrt{1-y^2}} = - \frac{dx}{\sqrt{1-x^2}}$

Integrating, we have $\sin^{-1} y + \sin^{-1} x = c$

$\Rightarrow \sin^{-1} [x\sqrt{1-y^2} + y\sqrt{1-x^2}] = c$

$x\sqrt{1-y^2} + y\sqrt{1-x^2}] = c$ is the required solution

8. Solve : $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$

Solution :

The given equation can be written as

$x\, e^x dx = \frac{-y}{\sqrt{1-y^2}}$

Integrating, we have

$\int x\, e^x \mathrm{dx} = -\int \frac{y}{\sqrt{1-y^2}} \mathrm{dy}$

$xe^x - \int e^x dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$ where $t = 1 - y^2$ so that $-2y\,dy = dt$

$\Rightarrow xe^x - e^x = \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$

$\Rightarrow xe^x - e^x = \sqrt{t} + c$

$\Rightarrow x\,e^x - e^x - \sqrt{1-y^2} = c$ which is the required solution.

9. Solve : $x\,dy = (y + 4x^5 e^{x^4})dx$

Solution :

$x\,dy - y\,dx = 4x^5\,e^{x^4}dx$

$\frac{xdy - ydx}{x^2} = 4x^3 e^{x^4} dx$

Integrating we have,

$\int \frac{xdy - ydx}{x^2} = \int 4x^3\,e^{x^4}\,\text{dx}$

$\int d\left(\frac{y}{x}\right) = \int e^t$ dtwhere $t = x^4$

$\Rightarrow \frac{y}{x} = e^t + c$

i.e., $\frac{y}{x} = e^{x^4} + c$ which is the required solution

10. Solve: $(x^2 - y)dx + (y^2 - x)\,dy = 0$, if it passes through the origin.

Solution :

$(x^2 - y)dx + (y^2 - x)\,dy = 0$

$x^2dx + y^2dy = xdy + ydx$

$x^2dx + y^2dy = d(xy)$

Integrating we have,

$\frac{x^3}{3}+\frac{y^3}{3}= xy + c$

Since it passes through the origin, $c = 0$

$\therefore$ the required solution is

$\frac{x^3}{3}+\frac{y^3}{3} = xy$ or $x^3 + y^3 = 3xy$

11. The normal lines to a given curve at each point (x, y) on the curve pass through the point (2, 0). The curve passes through the point (2, 3).Formulate the differential equation representing the problem and hence find theequation of the curve.

Solution :

Slope of the normal at any point $P(x, y) = -\frac{dx}{dy}$

Slope of the normal $AP = \frac{y-0}{x-2}$

$\therefore -\frac{dx}{dy} = \frac{y}{x-2}$ $\quad => x - 2 \Rightarrow\ = (2 - x)dx$

Integrating both sides, $\frac{y^2}{2} = 2x - \frac{x^2}{2} + c \qquad \cdots(1)$

Since the curve passes through (2, 3)

$\frac{9}{2} = 4 - \frac{4}{2} + c$

$c = \frac{5}{2}$

Put $c = \frac{5}{2}$ in (1),

$\frac{y^2}{2} = 2x - \frac{x^2}{2} + \frac{5}{2}$

$y^2 = 4x - x^2 + 5$

12. Solve: $\sec 2x\ dy - \sin 5x \sec^2 y\ dx = 0$

Solution:

The given equation can be written as

sec2x dy = sin5x sec^2y dx

$\frac{dy}{sec^2 y} = \frac{sin5x}{sec2x}$ dx

cos^2y dy= sin5x cos2x dx

Integrating we have

$\int \frac{1+cos2y}{2}$ dy $=\int \frac{sin7x+sin3x}{2}$ dx +c

⇨ y + $\frac{sin2y}{2}$ +$\frac{cos7x}{7}$ + $\frac{cos3x}{3}$ =c

13. Solve: cos^2 x dy + ye^{tanx} dx=0

Solution:

The given equation can be written as

$\frac{dy}{y} = \frac{-e^{tanx}}{cos^2x}$ dx

$\frac{dy}{y}$ = - e^{tanx} .sec^2x dx

Integrating we have

$\int \frac{dy}{y}$ = - $\int e^{tanx}$ .sec^2x dx

put t = tanx

dt = sec^2 x dx

log y + e^{tanx} = c

14. Solve: $(x^2$-y $x^2)$dy +$(y^2$ +x$y^2)$dx =0

Solution:

The given equation can be written as

$x^2(1-y)dy + y^2$ (1+x)dx =0

$$\frac{1-y}{y^2}\,dy + \frac{1+x}{x^2}\,dx = 0$$

Integrating we have

$$\int \frac{1}{y^2}\,dy - \int \frac{1}{y}\,dy + \int \frac{1}{x^2}\,dx + \int \frac{dx}{x} = c$$

$$\log x - \log y = \frac{1}{y} + \frac{1}{x} + c$$

$$\frac{x}{y} = e^{\left(\frac{x+y}{xy}\right)+c}$$

$$x = c\, y e^{\left(\frac{x+y}{xy}\right)+c}$$

15. Solve : y x^2 dx+ e^{-x} dy =0

Solution:

The given equation can be written as

$$y\,x^2\,dx = -\,e^{-x}\,dy$$

$$e^x x^2\,dx = -\frac{dy}{y}$$

Integrating,

$$\int e^x x^2\,dx = -\int \frac{dy}{y}$$

$u = x^2 \qquad dv = e^x\,dx$

$u' = 2x, \qquad v = e^x$

$u'' = 2 \qquad v_1 = e^x$

$v_2 = e^x$

$\int u\,dv = uv - u' v_1 + u'' v_2$

$e^x\,(x^2 - 2x + 2) + \log y = c$

16. $(x^2 + 5x + 7)dy + \sqrt{9 + 8y - y^2}\,dx = 0$

Solution:

The given equation can be written as

$\frac{dy}{\sqrt{9+8y-y^2}} = \frac{-dx}{x^2 + 5x + 7}$

Integrating,

$\int \frac{dy}{\sqrt{-[(y-4)^2 - 16 - 9]}} = \int \frac{dx}{(x+\frac{5}{2})2 - \frac{25}{4} + 7} + c$

$$\int \frac{dy}{\sqrt{5^{2-(y-4)^2}}} = -\int \frac{dx}{(x+\frac{5}{2})2 + (\frac{\sqrt{3}}{2})^2} + c$$

$\sin^{-1}\left(\frac{y-4}{5}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{5}{2}}{\frac{\sqrt{3}}{2}}\right) = c$

$$\sin^{-1}\left(\frac{y-4}{5}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+5}{\sqrt{3}}\right) = c$$

17. Solve: $\frac{dy}{dx} = sin(x+y)$

Solution:

Put z=x+y

Differentiating w.r.to. x

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

The given equation becomes

$$\frac{dz}{dx} = 1 + sinz$$

$$\frac{dz}{1+ sinz} = dx$$

$$\frac{1 - sinz}{1 - sin^2 z} = dx$$

Integrating,

$$\int \frac{1- sinz}{cos^2 z} dz = \int dx \quad + \text{c}$$

$$\int sec^2 z\, dz - \int tanz\, secz\, dz + \text{c}$$

$$tan(x+y) - \text{sec(x+y)} = \text{x+c}$$

18. Solve: y dx+ xdy=e^{-xy} dx if it cuts the y-axis

Solution:

The given equation can be written as

d(xy) =e^{-xy} dx

Integrating,

$\int e^{xy}$ d(xy) =$\int dx + c$

e^{xy} =x + c

Since it the y − axis , put x = 0 the we obtain c = 1

e^{xy} =x + 1

19. Solve :$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Solution :

Put $y = vx$

L.H.S. = v+ $x\frac{dv}{dx}$; R.H.S. = $v + \tan v$

$\therefore$ v+ $x\frac{dv}{dx}$= v+ tan vor $\frac{dx}{x} = \frac{cosv}{sinv}$ dv

Integrating, we have log$x$ = log sin v+ log c

$\Rightarrow x = c \sin v$

i.e., $x = c \sin\left(\frac{y}{x}\right)$

20. Solve : $xdy - ydx = x^2 + y^2 \, dx$

Solution :

From the given equation we have

$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \cdots (1)$

Put $y = vx$

L.H.S. = $v + x\frac{dv}{dx}$; R.H.S. $= \frac{v+\sqrt{1+v^2}}{1}$

$\therefore v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$ or $\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$

Integrating, we have, $\log x + \log c = \log [v + v^2 + 1]$

i.e., $xc = v + \sqrt{1+v^2}$

$\Rightarrow x^2 c = y + \sqrt{(y^2 + x^2)}$

21. $Solve: \frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

Solution:

$\frac{dy}{dx} = \frac{y^2}{x^2} - \frac{y}{x}$

$put\ y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = v^2 - v$

$x\frac{dv}{dx} = v^2 - 2v$

$\frac{dv}{v^2 - 2v} = \frac{dx}{x}$

$$\frac{dv}{(v-1)^2-1^2}=\frac{dx}{x}$$

$$\frac{1}{2}\ log\left[\frac{(v-1)-1}{(v-1)+1}\right]=logx\ +\ logc$$

$$\frac{1}{2}\ log\left[\frac{v-2}{v}\right]=logx\ +\ logc$$

$$\log\left[1-\frac{2}{v}\right]=\ 2logcx$$

$$1-\frac{2x}{y}=cx^2$$

$$(y-2x)=cx^2y$$

22. Solve: $\frac{dy}{dx}\ =\ \frac{y(x-2y)}{x(x-3y)}$

Solution:

Put $y=vx$

$$\frac{dy}{dx}\ =\ v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{x(x-2vx)}{x(x-3vx)}$$

$$v+x\frac{dv}{dx}\ \ =\ \frac{v-2v^2}{1-3v}$$

$$x\frac{dv}{dx}\ \ =\ \ \frac{v^2}{1-3v}$$

$$\frac{1-3v}{v^2}dv\ =\frac{dx}{x}$$

$(\frac{1}{v^2} - \frac{3}{v})dv = \frac{dx}{x}$

Integrating,

$-\frac{1}{v} - 3logv = logx + logc$

$-\frac{1}{v} = logv^3xc$

$-\frac{x}{y} = log\frac{y^3}{x^2}c$

$\frac{y^3}{x^2}c = e^{-x/y}$

$y^3 = c.x^2e^{-x/y}$

23. $(x^2 + y^2)dy = xydx$

Solution:

The given equation can be written as

$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

$put\ y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

$x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$

$$\frac{1+v^2}{v^3}dv = -\frac{dx}{x}$$

$$\left(\frac{1}{v^3}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating,

$$-\frac{1}{2v^2} + logv = -logx + logc$$

$$\log x + \log\frac{y}{x} - \log c = \frac{1}{2}\frac{x^2}{y^2}$$

$$\log\frac{y}{c} = \frac{1}{2}\frac{x^2}{y^2}$$

$$y = ce^{(x^2/2y^2)}$$

24. $x^2\frac{dy}{dx} = y^2 + 2xy \; giventhat \; y = 1, when \; x = 1$

Solution:

The given equation can be written as

$$\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$$

$put \; y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v^2 + 2v$$

$$x\frac{dv}{dx} = v^2 + v$$

$$\frac{dv}{v(v+1)} = \frac{dx}{x} \qquad \ldots\ldots\ldots..(1)$$

$$\frac{1}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1}$$

$A = 1, B = -1;$

$(1) \Rightarrow \left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \frac{dx}{x}$

$Integrating,$

$\log v - \log(v+1) = \log x + \log c$

$\log\left[\frac{v}{v+1}\right] = \log cx$

y=cx(x+y) $\quad (v=\frac{y}{x})$

It passes through (1,1)

1= c(1+1) => c=$\frac{1}{2}$

The solution is 2y= x(x+y)

25. Find the equation of the curve passing through (1,0) and which has slope $1+\frac{y}{x}$ at (x,y)

Solution:

The given equation can be written as

$\frac{dy}{dx} = 1+\frac{y}{x}$

$$\frac{dy}{dx} = \frac{x+y}{x}$$

$put\ y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$v + x\frac{dv}{dx} = \frac{x(1+v)}{x}$

$v + x\frac{dv}{dx} = 1+v$

$$dv = \frac{dx}{x}$$

Integrating,

v=log x + log c

$$\frac{y}{x} = logcx$$

y=x log cx

It is given that the curve passes through (1,0)

log c = 0

c=e^0 *=> c=1*

y=x log x

*26.*Solve : $\frac{dy}{dx} + y \cot x = 2 \cos x$

Solution :

The given equation is of the form $\frac{dy}{dx} + Py = Q$.

This is linear in *y*.

Here $P = \cot x$ and $Q = 2 \cos x$

I.F. = $e^{\int P dx}$

$= e^{\int \cot x \, dx}$

$= e^{\log \sin x}$

$= \sin x$

$\therefore$ The required solution is

y (I.F.) $= \int Q.(I.F)dx + c$

$\Rightarrow y (\sin x) = \int 2 \cos x \sin x \, dx + c$

$\Rightarrow y \sin x = \int \sin 2x \, dx + c$

$\Rightarrow y \sin x = -\frac{\cos 2x}{2} + c$

$\Rightarrow 2y \sin x + \cos 2x = c$

27. Solve : $(x + 1)\frac{dy}{dx} - y = e^x(x + 1)^2$

Solution :

The given equation can be written as

$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x + 1)$

This is linear in y.

Here $\int P dx = -\int \frac{1}{x+1} dx = -\log (x + 1)$

So I.F. $= e^{\int P dx} = e^{-\log(x + 1)} = \frac{1}{x+1}$

$\therefore$ The required solution is

$y\frac{1}{x+1} = \int e^x(x + 1)\frac{1}{x+1} dx + c$

$= \int e^x \, dx + c$

i.e., $\frac{y}{x+1} = e^x + c$

28. Solve : $\frac{dy}{dx} + 2y \tan x = \sin x$

Solution :

This is linear in y.

Here $\int P\, dx = \int 2\tan x\, dx = 2 \log \sec x$

I.F. $= e^{\int Pdx} = e^{\log \sec 2x} = \sec^2 x$

The required solution is

$y \sec^2 x = \int \sec^2 x \,.\, \sin x\, dx$

$= \int \tan x \sec x\, dx$

$\Rightarrow\ y \sec^2 x = \sec x + c$ or $y = \cos x + c \cos^2 x\, sec^2 x$

29. Solve: $\frac{dy}{dx} + y = x$

Solution:

The given equation is linear in y

P=1 ,Q=x

I.F.$=e^{\int pdx}$

$=e^{\int dx}$

$= e^{x}$

The required solution is

$ye^x = \int xe^x\, dx$

$= xe^x - \int e^x\, dx$

$=xe^{x} - e^{x} + c$

$ye^{x} = e^{x}(x-1) + c$

$e^{x}(y-x+1) = c$

29. Solve: $\frac{dy}{dx} + \frac{4x}{x^2+1}\, y = \frac{1}{(x^2+1)2}$

The given equation is linear in y

$P=\frac{4x}{x^2+1}$, $Q=\frac{1}{(x^2+1)2}$

I.F. $=e^{\int pdx}$

$=e\ 2\int \frac{2x}{x^2+1} dx$

$=e^{log\,(x^2+1)^2}$

$=(x^2+1)^2$

The required solution is

$y\,.(x^2+1)^2 = \int \frac{1}{(x^2+1)^2}(x^2+1)^2 dx$

$y\,.(x^2+1)^2 = x + c$

$y\,.(x^2+1)^2 - x = c$

30. Solve: $(1+x^2)\frac{dy}{dx} + 2xy = cosx$

Solution:

The given equation can be written as

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{cosx}{1+x^2}$$

$$P = \frac{2x}{1+x^2}, \quad Q = \frac{cosx}{1+x^2}$$

$$\text{I.F.} = e^{\int pdx}$$

$$= e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{log(1+x^2)}$$

$$= 1+x^2$$

Required equation is

$$y(1+x^2) = \int \frac{cosx}{1+x^2}(1+x^2)\,dx$$

$$y(1+x^2) = sinx + c$$

31. Solve: $\frac{dy}{dx} + xy = x$

Solution:

The given equation is linear in y

P= x Q=x

I.F.$=e^{\int p dx}$

$=e^{\int x dx}$

$=e^{\frac{x^2}{2}}$

Required solution is

$ye^{\frac{x^2}{2}} = \int xe^{(x^2/2)}dx$

put t $= \frac{x^2}{2}$ => x dx= dt

$ye^{\frac{x^2}{2}} = \int e^t\, dt$

$= e^t + c$

$ye^{\frac{x^2}{2}} = e^{\frac{x^2}{2}} + c$

$y=1+ c\, e^{-\frac{x^2}{2}}$

32. Solve: $(y - x)\frac{dy}{dx} = a^2$

Solution:

The given equation can be written as

$\frac{dx}{dy} = \frac{1}{a^2}y - \frac{1}{a^2}x$

$\frac{dx}{dy} + \frac{1}{a^2}x = \frac{1}{a^2}y$

This is linear in x

P=$\frac{1}{a^2}$ Q=$\frac{1}{a^2}y$

I.F.= $e^{\int pdx}$

=$e^{\frac{1}{a^2}\text{ dy}}$

=$e^{(y/a^2)}$

Required solution is

x$e^{(y/a^2)}$=$\int \frac{1}{a^2}$ y $e^{(y/a^2)}$dy

put t =$\frac{1}{a^2}y$ => $dy = a^2$ dt

x$e^{(y/a^2)}$ $= a^2 \int te^t\, dt$

$= a^2(te^t - e^t)$ +c

= a^2e^t (t-1) + c

$= a^2e^{(y/a^2)}\left[\frac{y}{a^2} - 1\right]$

$$\left[x - a^2\left(\frac{y}{a^2} - 1\right)\right]e^{(y/a^2)} = c$$

(x - y+a^2)$e^{(y/a^2)}$ $= c$

x = y - a^2 + $ce^{-(y/a^2)}$

33. Solve : $(D^2 - 13D + 12)y = e^{-2x}$

Solution :

The characteristic equation is $p^2 - 13p + 12 = 0$

$\Rightarrow (p - 12)(p - 1) = 0 \Rightarrow p = 12$ and 1

C.F. $= Ae^{12x} + B\,e^{x}$

Particular integral $P.I. = \frac{1}{D^2 - 13D + 12} e^{-2x}$

$$= \frac{1}{(-2)^2 - 13(-2) + 12} e^{-2x}$$

$$= \frac{1}{4+26+12} e^{-2x}$$

$$= \frac{1}{42} e^{-2x}$$

The general solution is y = CF + PI

$$y = Ae^{12x} + B\,e^{x} + \frac{1}{42} e^{-2x}$$

34. Solve: $(D^2 + 7D + 12)y = e^{2x}$

Solution:

The characteristic equation is $p^2 + 7p + 12 = 0$

$(p+4)(p+3) = 0$

p=-4 and p=-3

C.F.= $Ae^{-4x} + B\,e^{-3x}$

P.I. $= \frac{1}{D^2 + 7D + 12} e^{2x}$

P.I.= $\frac{e^{2x}}{4+14+12}$

$= \frac{e^{2x}}{30}$

The general solution is y=C.F.+P.I.

$$y= Ae^{-4x} + B\, e^{-3x} + \frac{e^{2x}}{30}$$

35. Solve: $(D^2 - 2D+13)y=e^{-3x}$

Solution:

The characteristic equation is

$P^2 - 4p+13 =0$

$P=\frac{4\pm\sqrt{16-52}}{2}$

$=2\pm 3i$

C.F. $= e^{2x}$(A cos3x +B sin3x)

P.I.$= \frac{1}{D^2-4D+13}e^{-3x}$

$= \frac{1}{9+12+13}e^{-3x}$

$= \frac{1}{34}e^{-3x}$

The general solution is y=C.F. +P.I.

$$y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{34}e^{-3x}$$

36. Solve: $(D^2 + 14D + 49)y = e^{-7x} +4$

Solution:

The characteristic equation is

$p^2+14p+49=0$

$(p+7)^2 = 0$

p=-7, -7

C.F.= (A x +B)e^{-7x}

PI$_1$= $\frac{1}{D^2+14D+49}e^{-7x}$

=$\frac{1}{(D+7)^2}e^{-7x}$

= $\frac{x^2}{2}e^{-7x}$

P.I$_2$= 4.$\frac{1}{D^2+14D+49}e^{0x}$

=$\frac{4}{49}$

The general solution is y= C.F. +P.I$_1$ + PI$_2$

$$y = (A\,x + B)e^{-7x} + \frac{x^2}{2}e^{-7x} + \frac{4}{49}$$

37. Solve: (D^2- 13D+12)y= $e^{-2x} + 5e^{x}$

Solution:

The characteristic equation is

p^2-13p+12 =0

(p-12)(p-1)=0

p=12, 1

C.F. = A e^{12x} + B e^{x}

PI$_1$ =$\frac{1}{D^2-13D+12}\ e^{-2x}$

= $\frac{1}{4+26+12}\ e^{-2x}$

$= \frac{e^{-2x}}{42}$

$PI_2 = \frac{1}{D^2-13D+12} 5e^x$

$= 5 \frac{1}{(D-12)(D-1)} e^x$

$= 5 \frac{1}{(-11)(D-1)} e^x$

$= \frac{-5}{11} xe^x$

The general solution is y=C.F. + PI_1 + PI_2

y= A e^{12x} + B e^x + $\frac{e^{-2x}}{42}$ - $\frac{5}{11}xe^x$

38. Solve: (D^2+1)y=0 when x=0, y=2 and when x= $\frac{\pi}{2}$,y= -2

Solution:

The characteristic equation is

$p^2+1=0$

$p=\pm i$

C.F.= A cos x+ B sin x

The solution is y= A cos x+ B sin x

When x=0, y=2 => A=2

When x= $\frac{\pi}{2}$,y= -2 => B=-2

The solution is y=2cosx-2sinx

y=2(cos x- sin x)

39. Solve: $(D^2+3D-4)y= x^2$

Solution:

The characteristic equation is

$p^2+3p-4=0$

$(p-1)(p+4)=0$

$p=1 ,-4$

$C.F.=Ae^{x}+B\,e^{-4x}$

$PI=c_0+c_1x+c_2x^2$

PI is also solution

$(D^2+3D-4)(c_0+c_1x+c_2x^2)=x^2$

$2c_2+ 3(c_1+2c2x) – 4(c_0+c_1x+c_2x^2)= x^2$

$2c_2+3c_1-4c_0=0$; $6c_2-4c_1=0$; $4c_2=1$

$\therefore c_2 = -\frac{1}{4}, c_1=\frac{-3}{8}, c_0= \frac{-13}{32}$

$PI= -\frac{1}{4}\left[x^2 + \frac{3x}{2} + \frac{13}{8}\right]$

The solution is y=CF+ PI

$$y= Ae^{x}+B\,e^{-4x} - \frac{1}{4}\left[x^2 + \frac{3x}{2} + \frac{13}{8}\right]$$

40. Solve: $(D^2-2D-3)y= \sin x \cos x$

Solution:

The characteristic equation is

$p^2-2p-3=0$

$(p-3)(p+1)=0$

$P=3, -1$

$CF=Ae^{3x}+ B\, e^{-x}$

$PI= \frac{1}{D^2-2D-3} \sin x \cos x$

$= \frac{1}{D^2-2D-3}\frac{1}{2} \sin 2x$

$= \frac{1}{2}\left[\frac{1}{D^2-2D-3}\right] \sin 2x$

$=-\frac{1}{2}\frac{2}{2D+7} \sin 2x$

$= -\frac{1}{2}\frac{2D-7}{4D^2-49} \sin 2x$

$= -\frac{1}{2}\frac{1}{-16-49} (2D-7) \sin 2x$

$= \frac{1}{130}(2.2\cos 2x-7\sin 2x)$

$= \frac{1}{130}(4\cos 2x-7\sin 2x)$

The general solution is y=CF +PI

$y= Ae^{3x}+ B\, e^{-x} + \frac{1}{130}(4\cos 2x - 7\sin 2x)$

41. Solve: $D^2y= -9\sin 3x$

Solution:

The characteristic equation is

$p^2=0$

$p= 0$ (twice)

$CF=(A x +B)e^{0x}$

$= (A x +B)$

$PI= \frac{1}{D^2}(-9 \sin 3x)$

$= -9 \frac{1}{D^2} \sin 3x$

$= \frac{-9}{-9}\sin 3x$ $\quad\therefore$ Replace D^2 by $- 3^2$

$=\sin 3x$

The general solution is $y=CF+ PI$

$$y= (Ax +B)+\sin 3x$$

42. Solve: $(D^2+5)y= \cos^2 x$

Solution:

The characteristic equation is

$p^2+5=0$

$p=\pm \sqrt{5}$

$CF= A \cos\sqrt{5}\, x +B \sin\sqrt{5}\, x$

PI= $\frac{1}{D^2+5}$ ($\cos^2$x)

PI= $\frac{1}{D^2+5}\left(\frac{1+\cos 2x}{2}\right)$

PI$_1$= $\frac{1}{2}\frac{1}{D^2+5}e^{0x}$ and PI$_2$= $\frac{1}{2}\frac{1}{D^2+5}$ cos2x

PI$_1$= $\frac{1}{2}\cdot\frac{1}{5}$

$= \frac{1}{10}$

PI$_2$= $\frac{1}{2}\frac{1}{D^2+5}$ cos2x

PI$_2$=$\frac{1}{2}\frac{1}{-4+5}$ cos2x   Replace D^2 by -2^2

$= \frac{1}{2}$cos2x

The general solution is y= CF + PI$_1$+PI$_2$

Y= A cos$\sqrt{5}$ x +B sin$\sqrt{5}$ x + $\frac{1}{10}+\frac{1}{2}$cos2x

43. Solve: $(D^2$+2D=3)y= sin2x

Solution:

The characteristic equation is

$p^2+2p+3=0$

p= $\frac{-2\pm\sqrt{4-12}}{2}$

$=-1\pm i\sqrt{2}$

CF = e^{-x}(A cos$\sqrt{2}x$ + $B\ sin\sqrt{2}x$)

$PI_1 = \frac{1}{D^2+2D+3}$ .sin2x

$= \frac{1}{-4+2D+3}$ sin2x   Replace D^2 by -2^2

$= \frac{2D+1}{4D^2-1}$ sin2x   Multiply and divide by 2D+1

$= \frac{(2D+1)}{-16-1}$ sin2x   Replace D^2 by -2^2

$= -\frac{1}{17}$ (4cos2x + sin2x)

The general solution is y=CF + PI

$$y= e^{-x}(A\cos\sqrt{2}x + B\sin\sqrt{2}x) - \frac{1}{17}(4\cos 2x + \sin 2x)$$

44. Solve: $(3D^2+4D+1)y=3e^{-x/3}$

Solution:

The characteristic equation is

$3p^2+4p+1=0$

$p= \frac{-4\pm\sqrt{16-12}}{6}$

$p= -1, -\frac{1}{3}$

$CF= Ae^{-x} + Be^{-x/3}$

$PI_1 = \frac{1}{3D^2+4D+1}\ 3e^{-x/3}$

$= 3\frac{1}{(3D+1)(D+1)}e^{-x/3}$

$= 3\frac{1}{3(D+\frac{1}{3})(D+1)}e^{-x/3}$

$= \frac{3}{2} \text{ x } e^{-x/3}$

The general solution is y= CF +PI

$$y = A\, e^{-x} + B\, e^{-x/3} + \frac{3}{2} \text{ x } e^{-x/3}$$

45. Solve : $(D^2 + 6D + 8)y = e^{-2x}$

Solution :

The characteristic equation is $p^2 + 6p + 8 = 0$

$\Rightarrow (p + 4)(p + 2) = 0$

$\Rightarrow p = -4$ and -2

The C.F. is $Ae^{-4x} + Be^{-2x}$

Particular integral $P.I. = \frac{1}{D^2+6D+8} e^{-2x}$

$$= \frac{1}{(D+4)(D+2)} \quad e^{-2x} \qquad \text{Since} f(D) = (D + 2)\ \theta(D))$$

$$= \frac{1}{\theta(-2)} \quad \text{x } e^{-2x}$$

$$= \frac{1}{2} \text{ x } e^{-2x}$$

Hence the general solution is $y = Ae^{-4x} + Be^{-2x} + \frac{1}{2} \text{ x } e^{-2x}$

46. Solve : $(D^2 - 6D + 9)y = e^{3x}$

Solution :

The characteristic equation is $p^2 - 6p + 9 = 0$

i.e., $(p - 3)^2 = 0$

$\Rightarrow p = 3, 3$

The C.F. is $(Ax + B)e^{3x}$

Particular integral $P.I. = \frac{1}{D^2-6D+9} e^{3x}$

$$= \frac{1}{(D-3)^2} e^{3x}$$

$$=\frac{x^2}{2}e^{3x}$$

Hence the general solution is $y = (Ax + B)e^{3x} + \frac{x^2}{2}e^{3x}$

47. Solve : $(2D^2 + 5D + 2)y = e^{-\frac{x}{2}}$

Solution :

The characteristic equation is $2p^2 + 5p + 2 = 0$

$$\therefore p = \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$$p = \frac{-1}{2}, p = -2$$

C.F.= $A\,e^{\frac{-x}{2}} + B\,e^{2x}$

P.I.= $\frac{1}{(2D2 + 5D + 2)}\,e^{\frac{-x}{2}}$

$$= \frac{1}{2\left(D+\frac{1}{2}\right)(D+2)}\,e^{\frac{-x}{2}}$$

$$= \frac{1}{\theta\left(\frac{-1}{2}\right).2} \text{ x } e^{\frac{-x}{2}}$$

$$= \frac{1}{3} \text{ x } e^{\frac{-x}{2}}$$

The general solution is y= $A\,e^{\frac{-x}{2}} + B\,e^{2x} + \frac{1}{3}\text{ x } e^{\frac{-x}{2}}$

48. Solve : $(D^2-4)y = \sin 2x$

Solution :

The characteristic equation is

$p^2-4 = 0$

$\Rightarrow p = \pm 2$

$C.F. = Ae^{2x} + Be^{-2x}$

P.I. = $\frac{1}{D^2-4}$ sin2x

$= \frac{1}{-4-4}$ sin2x

$= \frac{1}{-8}$ sin2x

The general solution is $y = C.F. + P.I.$

$$y = Ae^{2x} + Be^{-2x} - \frac{1}{8} \sin 2x$$

49. Solve : $(D^2 + 4D + 13)y = \cos 3x$

Solution :

The characteristic equation is

$p^2 + 4p + 13 = 0$

$p = \frac{-4 \pm \sqrt{16-52}}{2}$

$= \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm i6}{2}$

$= -2 \pm i3$

$C.F. = e^{-2x}(A \cos 3x + B \sin 3x)$

$P.I. = \frac{1}{D^2+4D+13} \cos 3x$

$= \frac{1}{-3^2+4D+13} \cos 3x$

$= \frac{1}{4D+4} \cos 3x$

$= \frac{(4D-4)}{(4D+4)(4D-4)} \cos 3x$

$= \frac{(4D-4)}{(16D^2-16)} \cos 3x$

$= \frac{(4D-4)}{-160} \cos 3x$

$= \frac{1}{40} (3 \sin 3x + \cos 3x)$

The general solution is y = C.F. + $P.I.$

$$y = e^{-2x}(A\cos 3x + B\sin 3x) + \frac{1}{40}(3\sin 3x + \cos 3x)$$

50. Solve $(D^2 + 9)y = \sin 3x$

Solution :

The characteristic equation is

$p^2 + 9 = 0$

$\Rightarrow p = \pm 3i$

$C.F. = (A\cos 3x + B\sin 3x)$

$P.I. = \frac{1}{D^2+9}\sin 3x$

$= \frac{-x}{6}\cos\ 3x \qquad \text{since } \frac{1}{D^2+a^2} = \frac{-x}{2a}\cos\ 3x$

Hence the solution is $y = C.F. + P.I.$

i.e., $y = (A\cos 3x + B\sin 3x) - \frac{-x}{6}\cos\ 3x$

51.Solve : $(D^2 - 3D + 2)y = x$

Solution :

The characteristic equation is

$p^2 - 3p + 2 = 0$

$\Rightarrow (p-1)(p-2) = 0$

$p = 1, 2$

The $C.F.$ is $(Ae^x + Be^{2x})$

Let $P.I. = c_0 + c_1x$

$\therefore c_0 + c_1x$ is also a solution.

$\therefore (D^2 - 3D + 2)(c_0 + c_1x) = x$

i.e., $(-3c_1 + 2c_0) + 2c_1x = x$

$\Rightarrow 2c_1 = 1$

$c_1 = \frac{1}{2}$

$(-3c_1 + 2c_0) = 0$

$\Rightarrow c_0 = \frac{3}{4}$

$\therefore$ P.I. $= \frac{x}{2} + \frac{3}{4}$

Hence the general solution is $y = C.F. + P.I.$

$y = Ae^{x} + Be^{2x} + \frac{x}{2} + \frac{3}{4}$

52. Solve : $(D^2 - 4D + 1)y = x^2$

Solution : The characteristic equation is

$p^2 - 4p + 1 = 0$

$\Rightarrow p = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

$C.F. = A\,e^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$

Let $P.I. = c_0 + c_1x + c_2x^2$

But P.I. is also a solution.

$\therefore (D^2 - 4D + 1)(c_0 + c_1x + c_2x^2) = x^2$

i.e., $(2c_2 - 4c_1 + c_0) + (-8c_2 + c_1)x + c_2x^2 = x^2$

$c_2 = 1$

$-8c_2 + c_1 = 0 \Rightarrow c_1 = 8$

$2c_2 - 4c_1 + c_0 = 0 \Rightarrow c_0 = 30$

P.I. $= x^2 + 8x + 30.$

Hence the general solution is $y = C.F. + P.I.$

$y = A\,e^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x} + x^2 + 8x + 30$

53. The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is constant temperature of surrounding medium. If initially $T = 150^\circ\ C$, find the temperature of the cooling object at any time t.

Solution :

Let T be the temperature of the cooling object at any time t

$\frac{dT}{dt} \alpha (T-S)]$

$\Rightarrow \frac{dT}{dt} = k(T-S)$

$\Rightarrow T - S = c\, e^{kt}$, where k is negative

$\Rightarrow T = S + ce^{kt}$

When $t = 0$, $T = 150$

$\Rightarrow 150 = S + c \Rightarrow c = 150 - S$

$\therefore$ The temperature of the cooling object at any time is

$T = S + (150 - S)e^{kt}$

For two subdivisions –Each 3 marks

1. Form the differential equations by eliminating arbitrary constants given in brackets against each

(i) $y^2 = 4ax$; {a}

Solution:

$y^2 = 4ax$ ....(1)

Differentiating w.r.to x

$$2y\frac{dy}{dx} = 4a$$

Substituting in equation (1)

$$y^2 = 2y.\frac{dy}{dx}.x$$

$$y = 2x\frac{dy}{dx}$$

(ii) $y = ax^2 + bx + c$; {a, b}

Solution:

The equation contains two arbitrary constants

$y = ax^2 + bx + c$ ....(1)

Differentiating twice w. r. to x

$\frac{dy}{dx} = 2ax + b$ ....(2)

$\frac{d^2y}{dx^2} = 2a$ ....(3)

$\Rightarrow y' = xy'' + b$

$= y' - xy''$

$(1) \Rightarrow y = x^2\left(\frac{y''}{2}\right) + x(y' - xy'') + c$

$y = -\frac{x^2}{2}y'' + xy' + c$

i.e., $x^2y'' - 2xy' + 2y - 2c = 0$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ {a, b}

Solution:

$b^2x^2 + a^2y^2 = a^2b^2$

Differentiating twice w.r.to. x

$2b^2x + 2a^2yy' = 0$ ...(1).

$b^2x + a^2yy' = 0$

$b^2 + a^2(y')^2 + a^2yy'' = 0$ ...(2)

$b^2 + a^2(y'^2 + yy'') = 0$

Eliminating a^2 and b^2 from(1) and (2) we get

$$\begin{vmatrix} x & yy' \\ 1 & y'^2 + yy'' \end{vmatrix} = 0$$

$x(y'^2 + yy'') - yy' = 0$

(iv) $y = Ae^{2x} + Be^{-5x}$ $\{A, B\}$

Solution:

$y = Ae^{2x} + Be^{-5x}$ ....(1)

Differentiating twice w.r.to. x

$y' = 2Ae^{2x} - 5Be^{-5x}$ ....(2)

$y'' = 4Ae^{2x} + 25Be^{-5x}$ ....(3)

Eliminating A and B between (1), (2) and (3)

$$\begin{vmatrix} y & e^{2x} & e^{-5x} \\ y' & 2e^{2x} & -5e^{-5x} \\ y'' & 4e^{2x} & 25e^{-5x} \end{vmatrix} = 0$$

$$e^{2x} . e^{-5x} \begin{vmatrix} y & 1 & 1 \\ y' & 2 & -5 \\ y'' & 4 & 25 \end{vmatrix} = 0$$

$$\begin{vmatrix} y & 1 & 1 \\ y' & 2 & -5 \\ y'' & 4 & 25 \end{vmatrix} = 0$$

$y(50+20) - y'(25-4) + y''(-5-2) = 0$

$y'' + 3y' - 10y = 0$

(v) y= (A +B x)e^{3x} {A,B}

Solution:

y= (A +B x)e^{3x}

Differentiate twice to eliminate two arbitrary constants

y' = (A+Bx)3e^{3x}+ e^{3x}.B

y' =3y + B e^{3x}

=>y'-3y = B e^{3x} ....(1)

y'' = 3y' +3B e^{3x}

$= 3y' + 3(y' - 3y)$ using(1)

y'' = 6y'-9y

$y'' - 6y'$+9y=0

(vi) y= $e^{3x}(C\ cos2x + D\ sin2x)${C,D}

Solution:

y= $e^{3x}(C\ cos2x + D\ sin2x)$ ......(1)

Differentiate twice to eliminate two arbitrary constants

$ye^{-3x} = C\ cos2x + D\ sin2x$

$ye^{-3x}(-3) + e^{-3x}y' = -2\ C\ cos2x + 2\ D\ sin2x$

$e^{-3x}(y' - 3y) = -2\ C\ cos2x + 2\ D\ sin2x$

Once again differentiate w.r.to. x

(y'' - 3y') – ($y' - 3y$) = - 4$e^{3x}(C\ cos2x + D\ sin2x)$

$y'' - 6y' + 9y = 0$

$y'' - 6y' + 13y = 0$

Ten marks questions

$1. (x + y)2\frac{dy}{dx} = 1 \quad \dots\dots(1)$

Solution:

Put z= x +y

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1$

$(1) \Rightarrow z^2\left(\frac{dz}{dx} - 1\right) = 1$

$z^2\frac{dz}{dx} = 1 + z^2$

$\frac{z^2}{1+z^2}\,dz = dx$

$\frac{(1+z^2-1)}{1+z^2}\,dz = dx$

$\int\left(1 - \frac{1}{1+z^2}\right)dz = \int dx + c$

$z - \tan^{-1}(x + y) = c$

$y - \tan^{-1}(x + y) = c \quad (z = x + y)$

2. Solve: $(x^2+y^2)dx+3xy\, dy = 0$

Solution:

The given equation can be written as

$$\frac{dy}{dx} = -\left(\frac{x^2+y^2}{3xy}\right) \quad ....(1)$$

Put $y = v\,x$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\left(\frac{x^2+v^2x^2}{3x^2v}\right)$$

$$v + x\frac{dv}{dx} = -\left(\frac{1+v^2}{3v}\right)$$

$$\frac{3v}{1+4v^2}\,dv = -\frac{dx}{x}$$

$$\frac{3}{8}\frac{8v}{1+4v^2}\,dv = -\frac{dx}{x}$$

$$\frac{3}{8}\log(1+4v^2) = -\log x + \log c$$

$$3\log(1+4v^2) + 8\log x = \log c$$

$$(1+4v^2)^3 . x^8 = c$$

$$\left(1 + 4\frac{y^2}{x^2}\right) . x^8 = c$$

$$(x^2+4y^2)^3\, x^2 = c$$

3. Solve: $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$

Solution:

The given equation is linear in x

$P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1} y}{1+y^2}$

I.F. $= e^{\int p dy}$

$= e^{\int (1/1+y^2) dy}$

$= e^{\tan^{-1} y}$

The required solution is

$x \tan^{-1} y = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}$ dy ......(1).

put $t = \tan^{-1} y$

$d\,t = \frac{1}{1+y^2}$ dy

(1) => $x \tan^{-1} y = \int t e^t \text{dt}$

$= te^t - \int e^t \text{dt}$

$= te^t - e^t + c$

$= e^t \ (t-1) + c$

$x \tan^{-1} y = e^{\tan^{-1} y} (\tan^{-1} y - 1) + c$

4. Solve: $\frac{dy}{dx} + \frac{y}{x} = \sin(x^2)$

Solution:

The given equation is linear in y

$P = \frac{1}{x}$ $\quad Q = \sin(x^2)$

I.F. $= e^{\int p dx}$

$= e^{\int \frac{1}{x} dx}$

$= e^{log x}$

$= x$

Required solution is

$y.x = \int sin(x^2)\, x\, dx$

put $t = x^2$

$dt = 2x dx$

$y.x = \int sint\ dt$

$= -\frac{1}{2} \cos t + c$

$= -\frac{1}{2} \cos x^2 + c$

$2xy + \cos x^2 = c$

5. Solve: dx+ x dy $=e^{-y}$

Solution:

The given equation can written as

$\frac{dx}{dy}+$ x= $e^{-y}\sec^2$y

The is linear in x

P=1 Q=$e^{-y}\sec^2$y

I.F. = $e^{\int pdx}$

$=e^{\int dy}$

$=e^{y}$

Required solution is

x $e^{y}=\int e^{-y}\sec^2 y e^{y}$ dy

$=\int \sec^2$y dy

x e^{y} $=$ $\tan y + c$

6. Show that the equation of the curve whose slope at any point is equal to y+2x and which passes through the origin is y= 2(e^{x}-x-1)

Solution:

$\frac{dy}{dx}$ =y+2x

$\frac{dy}{dx}$ –y =2x

This is linear in y

P=-1 Q= 2x

I.F.=$e^{\int -dx}$

=e^{-x}

Required solution is

y e^{-x}=$\int e^{-x}$.2x dx

=2$\{[-xe^{-x}] - \int -e^{-x}\, dx\}$ +c

= - 2x e^{-x} +2 $\int e^{-x}\, dx$ +c

y e^{-x}= - 2x e^{-x} $-$ $2e^{-x}$ +c

But the curve passes through (0,0)

0= -2+c

c= 2

y e^{-x} = - 2x e^{-x} $-$ $2e^{-x}$ +2

y= 2(e^{x}-x-1)

7. Solve : $\frac{d^2y}{dx^2}$ - 3y +2y =2e^{3x} when x=log2 , y=0 and when x=0, y=0

Solution:

The characteristic equation is

P^2-3p+2=0

(p-2)(p-1)=0

P=2, 1

C.F.= Ae^{2x} +B e^{x}

PI= $\frac{1}{D^2-3D+2}2e^{3x}$

$=2.\frac{1}{9-9+2}e^{3x}$

$=e^{3x}$

The general solution is y= CF + PI

$$y = Ae^{2x} + B\,e^{x} + e^{3x}$$

8. Solve: $(D^2$-6D+9)y =x + e^{2x}

Solution:

The characteristic equation is

$p^2-6p+9=0$

$(p-3)^2=0$

p= 3, 3

p=3(twice)

C.F.= (A +B x)e^{3x}

$PI_1= c_0+c_1x$

$(D^2-6D+9)(c_0+c_1x)=x$

$\therefore -6c_1 + 9(c_0 + c_1x) = x$

Equating coefficient of x and constant term

$c_1=\frac{1}{9}$ - $6c_1+9c_0=0$

$-6(\frac{1}{9}) + 9c_0 = 0$

$c_0 = \frac{2}{27}$

$PI_1 = \frac{x}{9} + \frac{2}{27}$

$PI_2 = \frac{1}{D^2-6D+9} e^{2x}$

$= \frac{1}{4-12+9} e^{2x}$

$= e^{2x}$

The general solution is y=C.F $+PI_1 + PI_2$

$y = (A + B x)e^{3x} + (\frac{x}{9} + \frac{2}{27}) + e^{2x}$

9. Solve: $(D^2 - 1)$y= cos2x – 2sin2x

Solution:

The characteristic equation is

$p^2 - 1 = 0$

$p = \pm 1$

C.F.=A e^x + B e^{-x}

$PI_1 = \frac{1}{D^2-1}$ (cos2x)

$= \frac{1}{-4-1}$ (cos2x) Replace D^2 by -2^2

$= -\frac{1}{5}$ cos2x

$PI_2 = \frac{1}{D^2-1}$ (-2sin2x)

$= -2\ \frac{1}{-4-1}$ (sin2x) Replace D^2 by -2^2

$= \frac{2}{5}$ sin2x

The general equation is $y = CF + PI_1 + PI_2$

$$y = A\,e^{x} + B\,e^{-x} - \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x$$

10. Radium disappears at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain at the end of 100 years.[Take A_0 be the initial amount].

Solution:

Let A be the amount of radium at time t

$$\frac{dA}{dt} \alpha A$$

$$\frac{dA}{dt} = kA$$

$A = ce^{kt}$

At t=0, $A = A_0$

$\therefore A_0 = ce^0$

$\therefore A = A_0\, e^{kt}$

But 5% of the original amount disappears in 50 years

When t=50, $A = 0.95A_0$

$\therefore 0.95A_0 = A_0 e^{50k}$

e^{50k}=0.95

Again , when t =100

A= $A_0 e^{100k}$

A= $A_0 e^{(50k)^2}$

=A_0 $(0.95)^2$

=0.9025A_0

The amount of radium that remains at the end of 100 years is 0.9025A_0

11. The sum of Rs. 1000 is compounded continuously, the nominal rate of interest being four percent per annum. In how many years will the amount be amount be twice the original principal?

Solution:

Let Abe the principal time t

$$\frac{dA}{dt}\alpha A$$

$$\frac{dA}{dt} = kA$$

$$\frac{dA}{dt} = \ 0.04t, \ \ since\ k = 0.04$$

A= c$e^{0.04t}$

When t=0, A=1000

$1000 = ce^0$

$c = 1000$

$\therefore A = 1000\, e^{0.04t}$

when A=2000, t=?

$2000 = 1000\, e^{0.04t}$

$t = \frac{\log 2}{0.04}$

$= \frac{0.6931}{0.04}$

= 17 years(app.)

12. A cup of coffee at temperature 100°C is placed in a room whose temperatureis 15°C and it cools to 60°C in 5 minutes. Find its temperature after a further interval of 5 minutes.

Solution:

Let T be the temperature of the coffee at any time t

By Newton's law of cooling,

$$\frac{dT}{dt}\, \alpha\, (T - S)$$

$$\frac{dT}{dt} = k\,(T - S)$$

$(T - S) = ce^{kt} \Rightarrow T = 15 + ce^{kt}$ since S=15°C

When t=0, T=100 => c=85

$$\therefore T = 15 + 85e^{kt}$$

When t=5, T=60

60= 15+85e^{5k}

$e^{5k} = \frac{45}{85}$

When t=10, T=?

T= 15+ 85 e^{10k}

=15+ 85 $(\frac{45}{85})^2$

= 38.82℃

13. The rate at which the population of a city increases at any time is proportional to the population at that time. If there were 1,30,000 people in the city in 1960 and 1,60,000 in 1990 what population may be anticipated in 2020

$\left[log(\frac{16}{13}) = 0.2070; e^{0.42} = 1.52\right]$

Solution:

Let A be the population at time t

$\frac{dA}{dt} \alpha A$

$\frac{dA}{dt} = kA$

A= ce^{kt}

Take the year 1960 as the initial time t=0

When t=0, A=1,30,000

130000= ce^0

C=130000

$\therefore A = 130000e^{kt}$

When the year 1990 i.e., when t=30, A=1,60,000

1,60000=130000e^{30k}

$e^{30k} = \frac{16}{13}$

When the year2020 i.e., when t=60, A=?

A=130000 e^{60k}

=130000$(\frac{16}{13})^2$

$\sim$ 197000

The approximate population in 2020 is 197000.

14. A radioactive substance disintegration at a rate proportional to its mass. When its mass is 10 mgm, the rate of disintegration is 0.051mgm per day. How long will it take for the mass to be reduced from 10 mgm to 5 mgm.(log2 =0.6931)

Solution:

Let A be the amount of mass at time t

$$\frac{dA}{dt} \alpha A$$

$$\frac{dA}{dt} = kA$$

A= ce^{kt}

When t=0, A=10 => c=10

A= 10 e^{kt}

Again $\frac{dA}{dt} = kA$

It is given that when A=10, $\frac{dA}{dt} = -0.051$ [$since\ \ disintegration$]

- 0.051 = 10k => k= -0.0051

A= $10e^{-0.0051t}$

When A=5

$5=10e^{-0.0051t}$

$\frac{1}{2} = e^{-0.0051t}$

$2= e^{0.0051t}$

log 2=0.0051t

t= $\frac{log\ 2}{0.0051}$

~ 136 days

The radioactive substance disintegrates 10 mgm to 5 mgm is 136 days

15. In a certain chemical reaction the rate of conversion of a substance at time *t* is proportional to the quantity of the substance still untransformed at that instant. At the end of one hour, 60 grams remain and at the end of 4 hours 21 grams. How many grams of the substance was there initially?

Solution :

Let *A* be the substance at time *t*

dA

dt α*A* ⇒*dA*

dt = *kA* ⇒*A* = ce^{kt}

When *t* = 1, *A* = 60 ⇒ce^{k}= 60 ⋯(1)

When $t = 4$, $A = 21 \Rightarrow ce^{4k} = 21 \cdots (2)$
$(1) \Rightarrow c^4 e^{4k} = 60^4 \cdots (3)$

$\frac{(3)}{(2)} \Rightarrow c^3 = \frac{60^4}{21} \Rightarrow c = 85.15$ (by using log)
Initially i.e., when $t = 0$, $A = c = 85.15$ gms (app.)
Hence initially there was 85.15 gms (approximately) of the substance

16. A bank pays interest by continuous compounding, that is by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year compounded continuously. Calculate the percentage increase in such an account over one year. [Take $e^{.08} \approx 1.0833$]

Solution : Let A be the principal at time t

$\frac{dA}{dt} \alpha A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{dt} = 0.08\,A$, since $k = 0.08$

$\Rightarrow A(t) = ce^{0.08t}$

Percentage increase in 1 year $= \frac{A(1) - A(0)}{A(0)}$

$$= \left(\frac{A(1)}{A(0)} - 1\right) \text{x}100$$

$$= \left(\frac{ce^{0.08}}{c} - 1\right) \text{x}100$$

$$= 8.33\%$$

Percentage increases is 8.33%